Problem 1. (20 points) Consider the differential equation $y' = y - xy^2$

5 (a) Sketch its direction field

5 (b) Given $y(x_0) = 0$ for some x_0, is it then true that $y(x) = 0$ for all real x? (Justify your answer)

5 (b) Find an appropriate substitution or change of variables so that the equation transforms into a linear or a separable equation

5 (c) Find explicitly all solutions of the given equation

(a)

Something like that

(b) Yes, by the exist. & uniqueness thm: since $y(x) = 0$ is a solution and $f(x,y) = y - xy^2$ continuous and $\frac{df}{dy} = 1 - 2xy$ also contin.

(c) Change $u = \frac{1}{y}$ (supp. $y(x) \neq 0$ for any x, see part (b))

then $u' = -\frac{y'}{y^2}$ $\Rightarrow (*)\ \underline{u' = x - u}$ Linear 1st order

(d) solving $(*)$ they should obtain $u = x - 1 + ce^{-x}$

and $y(x) = \frac{1}{x - 1 + ce^{-x}}$ or $y(x) \equiv 0$.

Problem 2. (20 points) A 100-gal tank initially contains 50 gal of pure water. Salt-water solution containing 0.5 lb salt for each gallon of water begins entering the tank at a rate of 5 gal/min. Simultaneously, a drain is opened at the bottom of the tank, allowing the salt-water to leave the tank at a rate of 3 gal/min. What is the salt content (lb) in the tank at the (first) moment when the tank is full?

$x(t)$ = lb of salt in TANK after t min.

$$\Rightarrow \frac{dx}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = 5\ \text{gal/min} \times 0.5\ \text{lb/gal} = \frac{5}{2}\ \text{lb/min}$$

$$\text{rate out} = 3\ \text{gal/min} \times \underbrace{\frac{x(t)}{50+2t}}_{\text{concentration}}\ \text{lb/gal}$$

$V(t)$ - volume at time t

$$\Rightarrow x' = \frac{5}{2} - \frac{3x}{2t+50}$$

linear eqn, can be solved:

$$x(t) = \frac{c}{\sqrt{(t+25)^3}} + t + 25 \qquad x(0) = 0 = \frac{c}{125} + 25$$

$$\Rightarrow c = -25 \cdot 125$$

Full TANK

$$50 + 2t = 100 \Rightarrow t = 25$$

$$x(25) = \frac{-25 \cdot 125}{\sqrt{50^3}} + 50 = \underbrace{50 - \frac{25}{2\sqrt{2}}}_{\text{fine as answer}} \approx 41.16\ \text{lb}$$

Problem 3. (20 points) Consider the differential equation $(xy-1)dx+(x^2-xy)dy=0$

5 (a) Show that it is not exact

10 (b) Find its integrating factor $\mu = \mu(x)$ if it depends on x only

10 (c) Find its general solution using the integrating factor μ

$$\underbrace{(xy-1)}_{P}dx+\underbrace{(x^2-xy)}_{Q}dy=0$$

(a) If it's exact then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, but $\frac{\partial P}{\partial y} = x$, $\frac{\partial Q}{\partial x} = 2x-y$

(b) By the method they can find that $\boxed{\mu(x)=\frac{1}{x}}$ works

(c) then $\mu(Pdx+Qdy)$ is exact and conclude that $F(x,y)=xy-\ln x-\frac{y^2}{2}=C$ is an implicit solution and then the general sol'n can be obtained as

$$y = x \pm \sqrt{x^2-2\ln x+\tilde{C}}$$

Problem 4. (20 points) Consider the autonomous equation $y' = (y-1)^2(y+1)$

5 (a) Find its equilibrium points

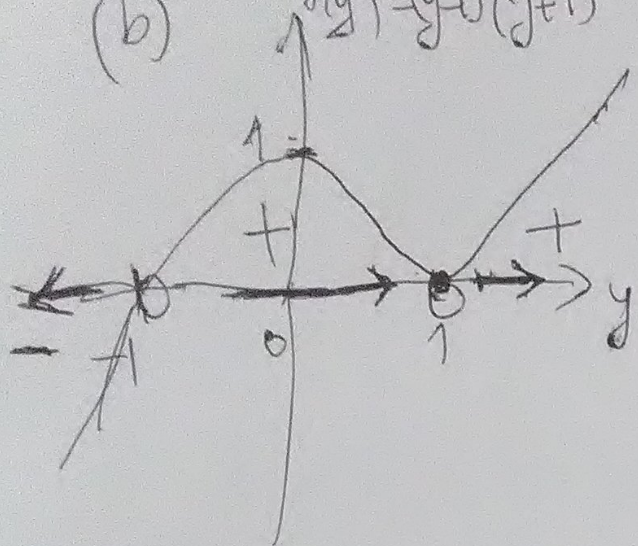
5 (b) Draw a phase diagram and describe asymptotically stable and unstable points

5 (c) Sketch the equilibrium solutions in the xy-plane. These solutions divide the plane into regions. Sketch at least one solution trajectory in each of these regions.

5 (d) Find $\lim_{x\to\infty} y(x)$ for the solution y satisfying the initial condition $y(0) = 0.5$.

(a) $y=1, -1$ equilibrium points & solutions

(b)

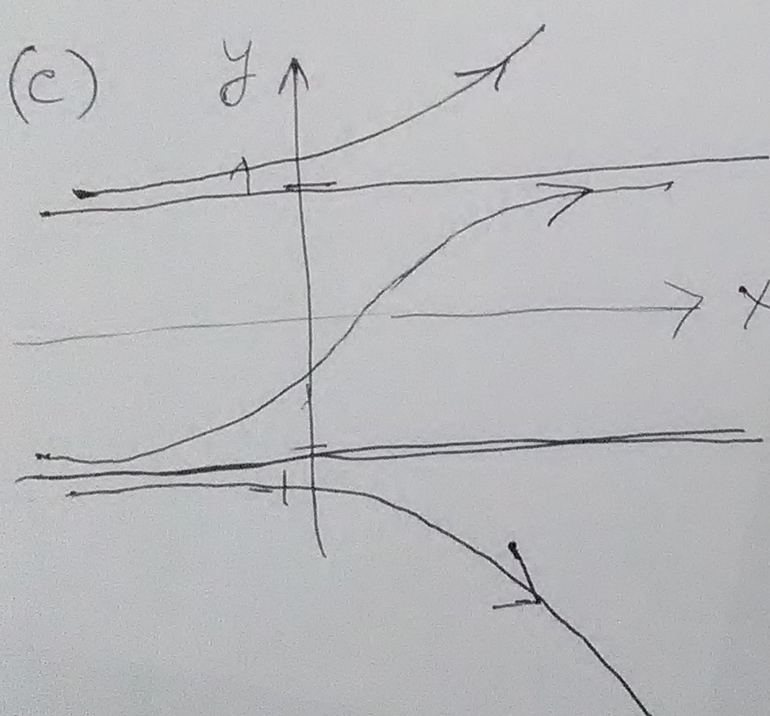


1 is local ~~min~~ & f not decreasing

−1, 1 unstable equilibrium points.

x_0 is stable in $f'(x_0)<0$
i.e. decreasing at x_0

(c)



(d) if $y(0)=0.5$
it's between $(-1,1)$
and $y(x)$ ~~sho~~ approaches 1 as $x \to \infty$.

$\lim_{x\to\infty} y(x) = 1$.