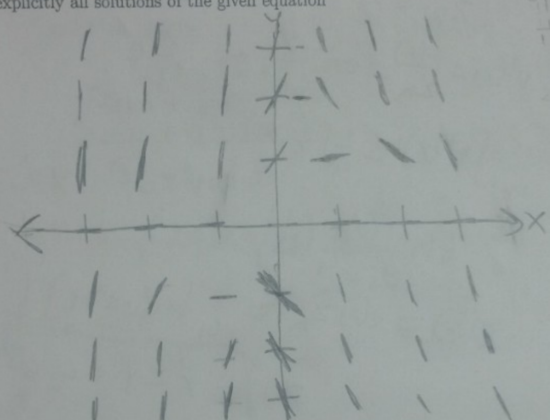
Problem 1. (20 points) Consider the differential equation $y' = y - xy^2$
(a) Sketch its direction field
(b) Given $y(x_0) = 0$ for some x_0, is it then true that $y(x) = 0$ for all real x? (Justify your answer)
(b) Find an appropriate substitution or change of variables so that the equation transforms into a linear or a separable equation
(c) Find explicitly all solutions of the given equation

a.)



b.) yes, if $y(x_0)=0$ $\quad y' = 0 - x_0(0) = 0$ for all X, then if $y' = 0$ for all x & $y(x_0)=0$, the function will stay at 0

why?

c.) where $n=2$

$z = y^{1-n}$ $\quad z' = (1-n)\,y^{-n}\,y'$ $\quad y' = \frac{z'y^n}{(1-n)}$

$y' = y - xy^2$

$\frac{z'y^n}{(1-n)} = y - xy^2$

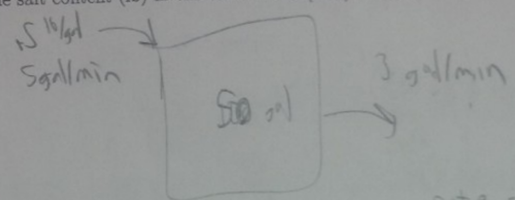
$z' = (y^{1-n} - xy^{2-n})(1-n)$

$n=2$ $\quad z' = (z - xy^0)(1-n)$

$z' = (z - x)(-1) = x - z$

$\boxed{z' = x - z}$

Problem 2. (20 points) A 100-gal tank initially contains 50 gal of pure water. Salt-water solution containing 0.5 lb salt for each gallon of water begins entering the tank at a rate of 5 gal/min. Simultaneously, a drain is opened at the bottom of the tank, allowing the salt-water to leave the tank at a rate of 3 gal/min. What is the salt content (lb) in the tank at the (first) moment when the tank is full?



rate = conc. x vol rate

conc. = lbs salt / gal water

$\frac{dc}{dt}$ = rate in − rate out

$= (.5\,\text{lb/gal}) \times (5\,\text{gal/min}) - \frac{c(t)}{50+2t} \times 3\,\text{gal/min}$

$v(t) = 50 + 5t - 3t$

$100 = 50 + 2t \quad t = 25$ tank is full

$c' = \frac{5}{2} - \frac{3c}{50+2t}$ ✓

$c_h' = \frac{-3c_h}{50+2t}$

$\int \frac{c_h'}{c_h} = \int -\frac{3}{50+2t}$

$\ln(c_h) = \frac{-3 \cdot \ln(50+2t)}{2} + C \qquad c_h = e^{\ln(50+2t)^{-3/2} + C} = e^C e^{\ln(50+2t)^{-3/2}}$

$c_h = \frac{c_1}{(50+2t)^{3/2}}$

$v'(t) = \frac{f(t)}{c_h} = \frac{\frac{5}{2}}{\frac{c_1}{(50+2t)^{3/2}}} = \frac{5(50+2t)^{3/2}}{2c_1} \qquad \int v'(t) = v(t) = \frac{(50+2t)^{5/2}}{2c_1}$

$v(t) = \frac{(50+2t)^{5/2}}{2c_1} + c_2$

$c(t) = v(t)\,c_h = \left(\frac{(50+2t)^{5/2}}{2c_1} + c_2\right)\left(\frac{c_1}{(50+2t)^{3/2}}\right)$

$c(t) = \frac{(50+2t)}{2} + \frac{c_3}{(50+2t)^{3/2}}$

$c(0) = 0 = 25 + \frac{c_3}{(50)^{3/2}}$

$c_3 = -25(50)^{3/2}$

$c(25) = \frac{100}{2} - \frac{25(50)^{3/2}}{(100)^{3/2}} = 50 - 25\left(\frac{1}{2}\right)^{3/2} = \boxed{50 - 25\left(\frac{\sqrt{2}}{2}\right)^{3}\,\text{lb}}$ ✓

Problem 3. (20 points) Consider the differential equation $(xy-1)dx + (x^2 - xy)dy = 0$

(a) Show that it is not exact

(b) Find its integrating factor $\mu = \mu(x)$ if it depends on x only

(c) Find its general solution using the integrating factor μ

a) $P = xy - 1 \quad Q = x^2 - xy$

$\frac{\partial P}{\partial y} = x \quad \frac{\partial Q}{\partial x} = 2x - y \quad \boxed{\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \text{ not exact}}$ V

b) $g = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{x^2 - xy}(x - 2x + y) = \frac{1}{-x(y-x)}(y-x)$

int factor $= e^{\int g(x)} = e^{\int -\frac{1}{x}} = e^{-\ln x} = e^{\ln x^{-1}} = \boxed{\frac{1}{x} = \mu}$ V

c) $\mu P = y - \frac{1}{x} \quad Q = x - y$

$\frac{\partial P}{\partial y} = 1 \quad \frac{\partial Q}{\partial x} = 1$

$\int P dx = xy - \ln x + C \quad \int Q dy = xy - \frac{y^2}{2} + C$ V

$F(x,y) = C \quad \boxed{F(x,y) = xy - \frac{y^2}{2} - \ln x = C}$

$y = ?$

Problem 4. (20 points) Consider the autonomous equation $y' = (y-1)^2(y+1)$

(a) Find its equilibrium points

(b) Draw a phase diagram and describe asymptotically stable and unstable points

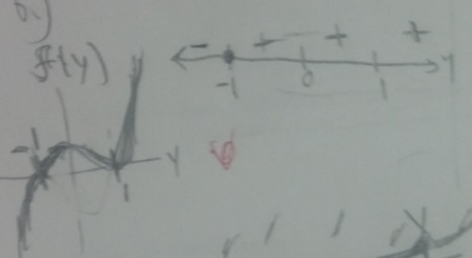
(c) Sketch the equilibrium solutions in the xy-plane. These solutions divide the plane into regions. Sketch at least one solution trajectory in each of these regions.

(d) Find $\lim_{x\to\infty} y(x)$ for the solution y satisfying the initial condition $y(0) = 0.5$.

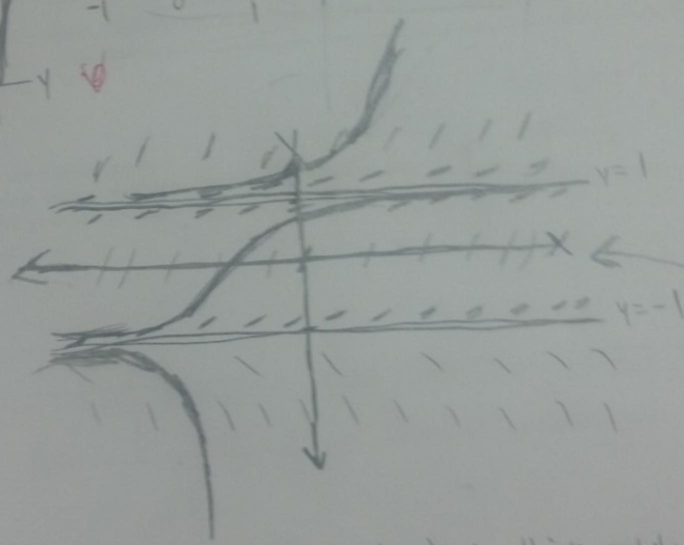
a) $y' = 0 = (y-1)^2(y+1)$ $\boxed{y = 1, -1}$ ✓

b) $y' = (y-1)(y-1)(y+1)$

for $(y, f(y))$, $(-1,0)$ & $(1,0)$ are unstable points ✓



c)



d) $y(0) = .5$ $-1 < .5 < 1$ which is this solution

therefore $\lim_{x\to\infty} y(x)$ where $y(0) = .5$ is 1 ✓