W. Conley | Math 33B, Lecture 1 | Fri, Apr 24, 2015

Midterm 1

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Section: Tuesday: Thursday:

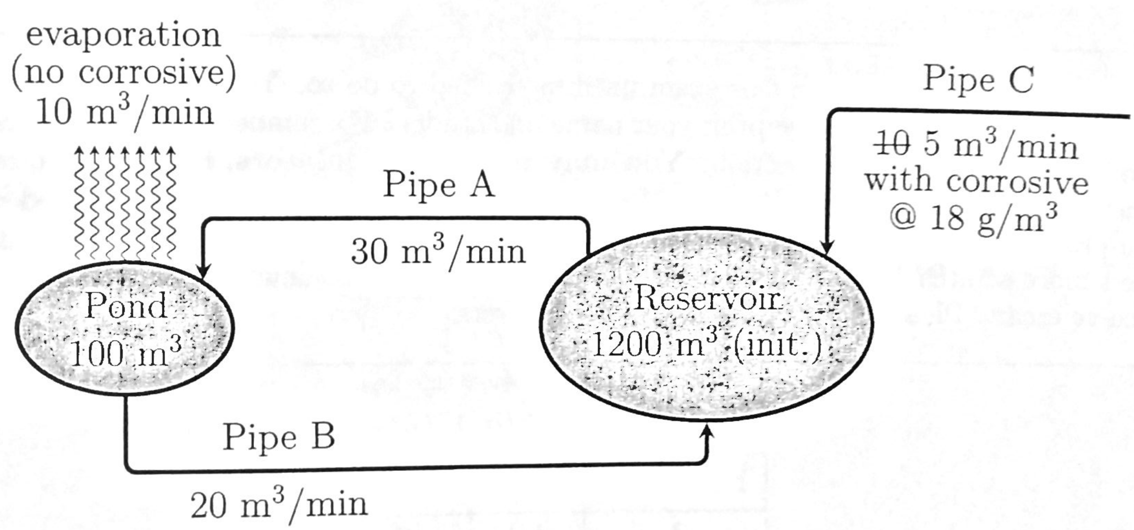
Tuesday	Thursday	
1A	1B	TA: Wei Zhu
1C	1D	TA: Paul (Yuming) Zhang
1E	1F	TA: Josh Cutler

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators**, books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Please do not write below this line.

Problem	Max	Score
1	10	10
2	10	10
3	10	10
4	8	8
5	7	7
Total	45	45

1. (10 pts) At a nuclear power plant, spent fuel rods are kept in a small cooling pond containing 100 m^3 of pure water. Due to the heat produced by these fuel rods, the water in this pond evaporates at a rate of 10 m^3/min. Since it is essential that this pond be kept full and relatively cool, this water is exchanged with water from a larger reservoir outside the plant: water is pumped from that reservoir to the pond through pipe A at 30 m^3/min, and pumped from the bottom of the pond back to the reservoir through pipe B at 20 m^3/min. The reservoir initially contains 1200 m^3 of pure water. To keep the water level in this reservoir constant, water is supposed to be pumped into it through pipe C at a rate of 10 m^3/min. However, an earthquake (at time $t = 0$) causes pipe C's pump to malfunction, so that it only operates at 5 m^3/min. And to make matters worse, the earthquake also causes a crack in this pipe, which introduces a corrosive at a concentration of 18 g/m^3. Assume that this corrosive will not evaporate with the water in the cooling pond.



Set up a system of differential equations *(with initial conditions)* to model the amount of the corrosive in the reservoir and in the pond. Note that you should have 2 or 3 variables. Be sure to state clearly what each of your variables represents.

let c = grams of corrosive

C_R = reservar corrosive

t = minutes

C_P = pond corrosive

$V_R = 1200 - 5t$

$$\frac{dC_R}{dt} = 5 \cdot 18 - 30 \cdot \frac{C_R}{1200-5t} + 20 \cdot \frac{C_P}{100} = 90 - \frac{6C_R}{240-t} + \frac{C_P}{5}$$

$C_R(0) = 0$

$$\frac{dC_P}{dt} = \frac{30\,C_R}{1200-5t} - \frac{20C_P}{100} = \frac{6C_R}{240-t} - \frac{C_P}{5}$$

$C_P(0) = 0$

2. (10 pts) Solve the initial value problem

$$(y^2+4)\sin(x) + 2y\cos(x)y' = 0 \qquad y(\tfrac{\pi}{3}) = 1.$$

not linear, is separable

∂

$\tan(x) + \frac{2y}{y^2+4} y' = 0$

$\frac{\partial (y^2+4)\sin x}{\partial y} - \frac{\partial\, 2y\cos x}{\partial x}$

$2y\sin x + 2y\sin x = 4y\sin x$

$\int \frac{2y}{y^2+4} y' dx = \int -\tan(x)\, dx$

$u = e^{-\int a t}$

$\int \frac{2y}{y^2+4} dy = \int -\tan x\, dx$

$\int \frac{1}{y^2+4} d(y^2+4) = \int -\tan x\, dx$

$\ln \cos x$

$\ln|y^2+4| = \ln|\cos x| + C$

$e^C e^{\ln\cos x}$

$\frac{1}{\cos x} \quad \sin x$

$y^2+4 = A\cos x$

$-\frac{\sin x}{\cos x}$

$y = \sqrt{A\cos x - 4}$

$y(\frac{\pi}{3}) = 1 = \sqrt{A\cos\frac{\pi}{3} - 4} \Rightarrow 5 = A/2 \Rightarrow A = 10$

$\boxed{y(x) = \sqrt{10\cos x - 4}}$

$y' = \frac{-10\sin x}{2\sqrt{10\cos x - 4}}$

$10\cos x \sin x - 10\sin x\cos x = 0$ ✓

3. (10 pts) At a barbecue, you leave a cold bottle of beer outside, initially at 40° F. From Newton's Law of Cooling, you know that the rate of change of the bottle's temperature will be proportional to the difference between the its current temperature (T) and the temperature of the air around it (A), and from past experience, you have calculated that the proportionality constant for this brand of beverage is 70% per hour (i.e., 0.7). The air outside is 72° F initially, but it is warming up at 2° F per hour. Set up and solve this differential equation, to find a function for the temperature of the bottle at any time t.

$$\frac{dT}{dt} = -k(T-A) = -0.7(T-(72+2t))$$

$$= -0.7T + 50.4 + 1.4t$$

$$0.7T + T' = 50.4 + 1.4t \qquad u = e^{\int 0.7\,dt} = e^{0.7t}$$

$$\int (0.7e^{0.7t}T + e^{0.7t}T')\,dt = \int (50.4e^{0.7t} + 1.4te^{0.7t})\,dt$$

$u = t \quad du = dt$

$dv = ke^{kt}dt \quad v = e^{kt}$

$\int kte^{kt} = te^{kt} - \int e^{kt}dt$

$= te^{kt} - \frac{1}{k}e^{kt}$

$$e^{0.7t}T = 72e^{0.7t} + 2te^{0.7t} - \frac{20}{7}e^{0.7t} + C$$

$$T = 72 + 2t - \frac{20}{7} + \frac{C}{e^{0.7t}} = 2t + 69\tfrac{1}{7} + \frac{C}{e^{0.7t}}$$

$$T(0) = 40 = 69\tfrac{1}{7} + C \Rightarrow C = -29\tfrac{1}{7}$$

$210 - 6 = 204$

$$\boxed{T(t) = 69\tfrac{1}{7} + 2t - (29\tfrac{1}{7})e^{-0.7t}}$$

$490 - 6$

$$T' = 2 + \frac{102}{5}e^{-0.7t} = -0.7\left(\frac{484}{7} + \cancel{2t} - \cancel{\frac{204}{7}e^{-0.7t}}\right) + 50.4 + \cancel{1.4t}$$

$$2 = \frac{504}{10} - \frac{484}{10} \checkmark$$

10/10

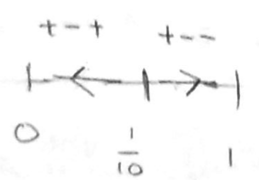
4. (8 pts) A scientist has introduced a genetic mutation into a population of mice that he is studying in his lab. He asks you to model the spread of the mutation through the population, as they breed normally over the next several generations. You come up with the following model, where $p(t)$ is the *fraction* of mice that carry the genetic mutation:

$$p' = p(p-1)(1-10p)$$

If the scientist originally introduced the mutation into 13% of the mice in his lab, then in the long run (as $t \to \infty$) what fraction of the mice will end up with the mutation? *(Hint: You do not need to solve this differential equation!)*

equilibrium points $p=0$, $p=1$, $p=\frac{1}{10}$

p goes from 0 to 1



$p' = .13(-.87)(-0.3)$

as $t \to \infty$ with a p in between $\frac{1}{10}$ and 1

$\boxed{p \to 1}$

5. (7 pts) Consider the differential equation

$$(t+2)y' = y^{\frac{2}{3}}.$$

(a) For what points (t_0, y_0) does the Existence Theorem guarantee that a solution exists satisfying $y(t_0) = y_0$?

$$y' = \frac{y^{\frac{2}{3}}}{t+2}$$

For any point (t_0, y_0) where $t \neq -2$, there is a guaranteed solution satisfying $y(t_0) = y_0$.

(b) For what points (t_0, y_0) does the Uniqueness Theorem guarantee that there is only one solution satisfying $y(t_0) = y_0$?

$$F(y,t) = y' = \frac{y^{\frac{2}{3}}}{t+2}$$

$$\frac{\partial F}{\partial y} = \frac{2}{3} y^{-\frac{1}{3}} (t+2)^{-1}$$

For any point (t_0, y_0) where $y \neq 0$ and $t \neq -2$ there is only one solution satisfying $y(t_0) = y_0$