W. Conley | Math 33B, Lecture 1 | Fri, May 15, 2015

Midterm 2

Last Name: ______________________

First Name: ______________________

Student ID: ______________________

Signature: ______________________

Section: Tuesday: Thursday:

TA: Wei Zhu

TA: Paul (Yuming) Zhang

TA: Josh Cutler

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Please do not write below this line.

Problem	Max	Score
1	10	9
2	10	10
3	10	10
4	10	8
Total	~~45~~	37

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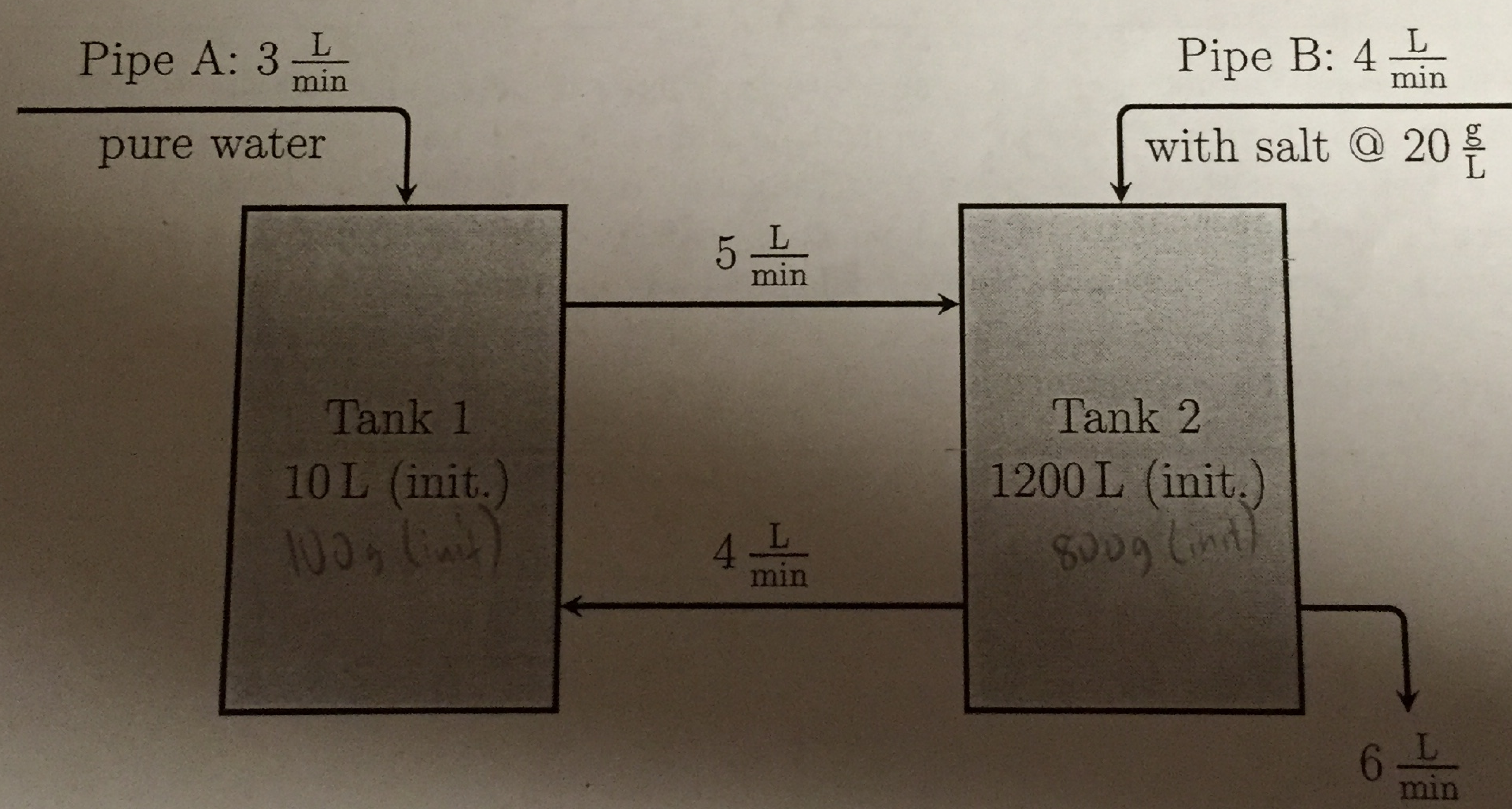
1. (10 pts) Tank 1 initially contains 10 L of water with 100 g of salt. Tank 2 initially contains 1200 L of water with 800 g of salt. The tanks are connected by pipes with flow rates as shown in the diagram below. Note that the volume of solution in each tank is changing! Note also that the water entering tank 1 through pipe A is pure water, whereas the solution entering tank 2 through pipe B contains salt at a concentration of 20 $\frac{g}{L}$. (In all other pipes, the concentration of salt is of course equal to the concentration in the tank that the pipe is coming from.)

Letting y_1 and y_2 be the amounts of salt (in grams) in tank 1 and tank 2 respectively, and using these (and t) as your *only* variables, **write down a system of differential equations, with initial conditions,** to model this situation.

Is the resulting system linear? If so, write it in matrix form, and also specify if it is homogeneous, and whether or not it has constant coefficients.



(Answer on the following page.)

(Problem 1 continued...)

$$y_1' = -5\frac{L}{min} \cdot \frac{y_1 g}{V_1(t)} + 4\frac{L}{min} \cdot \frac{y_2}{V_2(t)}$$

$$V_1(t) = 10 + 3t + 4t - 5t = 10 + 2t$$

$$V_2(t) = 1200 + \cancel{4t} + 5t - \cancel{4t} - 6t = 1200 - t$$

$$y_2' = 4\frac{L}{min} \cdot \frac{20g}{L} + 5\frac{L}{min} \cdot \frac{y_1 g}{V_1(t)} - 10\frac{L}{min} \cdot \frac{y_2}{V_2(t)}$$

$$y_1' = -\frac{5y_1}{10+2t} + \frac{4y_2}{1200-t}$$

$$y_2' = \frac{5y_1}{10+2t} - \frac{10y_2}{1200-t} + 80$$

$$\boxed{\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} \frac{-5}{10+2t} & \frac{4}{1200-t} \\ \frac{5}{10+2t} & \frac{-10}{1200-t} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 80 \end{bmatrix}}$$

l.l.?

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Linear? YES NO

Homogeneous? YES NO

Constant coefficients? YES NO

2. (10 pts) Find the general solution of the system

$$\mathbf{y}' = \begin{bmatrix} -7 & -4 \\ 4 & 1 \end{bmatrix} \mathbf{y}.$$

$A - \lambda I_2 = \begin{bmatrix} -7-\lambda & -4 \\ 4 & 1-\lambda \end{bmatrix}$

$\det(A - \lambda I_2) = 0$

$(-7-\lambda)(1-\lambda) + 16 = 0$

$-7 - \lambda + 7\lambda + \lambda^2 + 16 = 0$

$\lambda^2 + 6\lambda + 9 = 0$

$(\lambda+3)^2 = 0$

$\lambda = -3$

$\ker\begin{pmatrix} -7--3 & -4 \\ 4 & 1--3 \end{pmatrix} = \ker\begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \to x_1 = -x_2$

$\to \begin{bmatrix} -t \\ t \end{bmatrix} \to t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$y(t) = C_1 e^{-3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{-3t}(v_2 + t v_1)$ where $(A - \lambda I_2)v_2 = v_1$

$\begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to \left[\begin{array}{cc|c} -4 & -4 & -1 \\ 4 & 4 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 1/4 \\ 1 & 1 & 1/4 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 1 & 1/4 \\ 0 & 0 & 0 \end{array}\right] \to x_1 + x_2 = 1/4 \to x_1 = -x_2 + 1/4 \to \begin{bmatrix} -t+1/4 \\ t \end{bmatrix}$

Let $t = 1$, $\begin{bmatrix} -1+1/4 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix} = v_2 \to \begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -t-3/4 \\ 1+t \end{bmatrix}$

$-\frac{5}{4}$

$$\boxed{y(t) = C_1 e^{-3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{-3t}\begin{bmatrix} -t-3/4 \\ 1+t \end{bmatrix}}$$

3. (10 pts) Find the general solution of the system

$$\mathbf{y}' = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix} \mathbf{y}.$$

$\det(A - \lambda I_2) = 0$

$(5-\lambda)(3-\lambda) + 2 = 0$

$15 - 8\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 8\lambda + 17 = 0$

$$\frac{8 \pm \sqrt{64 - 4(17)}}{2} = \frac{8 \pm \sqrt{64-68}}{2} = \frac{8 \pm \sqrt{-4}}{2} = \frac{8 \pm 2i}{2} = 4 \pm i$$

$= 4+i, \; \lambda_2 = 4-i$

$\ker \begin{pmatrix} 5-(4+i) & 2 \\ -1 & 3-(4+i) \end{pmatrix} = \ker \begin{pmatrix} 1-i & 2 \\ -1 & -1-i \end{pmatrix}$

$\begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1+i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow y_1(t) = e^{(4+i)t} \cdot \begin{bmatrix} 2 \\ -1+i \end{bmatrix} = e^{4t} e^{it} \begin{bmatrix} 2 \\ -1+i \end{bmatrix}$

$-2+2i = 0$

$(-1-i)(-1+i) = -2+1+1 = 0$

$$= e^{4t}(\cos t + i \sin t) \begin{bmatrix} 2 \\ -1+i \end{bmatrix}$$

$$= e^{4t}(\cos t + i \sin t)\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= e^{4t}\left(\cos t \begin{bmatrix} 2 \\ -1 \end{bmatrix} + i \cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \sin t \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= e^{4t}\left(\begin{bmatrix} 2\cos t \\ -\cos t - \sin t \end{bmatrix} + i \begin{bmatrix} 2 \sin t \\ \cos t - \sin t \end{bmatrix}\right)$$

$$y(t) = C_1 e^{4t} \begin{bmatrix} 2\cos t \\ -\cos t - \sin t \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} 2 \sin t \\ \cos t - \sin t \end{bmatrix}$$

$1+\lambda+\lambda+\lambda^2+16=0$

$x^2+2x+17=0$

$\frac{-2\pm\sqrt{4-4(17)}}{2}$ = -2

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4. (10 pts) Match each of the following systems of differential equations with the phase portraits on the next page, and classify the type of equilibrium point at the origin for each one.
(1 pt each)

Linear System	Plot	Type of Equilibrium	
$\mathbf{y}' = \begin{bmatrix} -4 & 1 \\ -2 & -1 \end{bmatrix} \mathbf{y}$	E	Stable	
$\mathbf{y}' = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \mathbf{y}$	D	Saddle ✓ (generally unstable)	
$\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ -4 & -1 \end{bmatrix} \mathbf{y}$	✓ C	unstable spiral	✗
$\mathbf{y}' = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \mathbf{y}$	A	unstable	
$\mathbf{y}' = \begin{bmatrix} -1 & 8 \\ -2 & -1 \end{bmatrix} \mathbf{y}$	B	stable spiral	✗

$(3-\lambda)(-1-\lambda)+8=0$

$-3+\lambda-3x+\lambda^2+8=0$

$\lambda^2-2\lambda+5=0$

$(3-\lambda)(1-\lambda)-8=0$

$3-\lambda-3\lambda+\lambda^2-8=0$

$\lambda^2-4\lambda-5=0,\ (\lambda-5)(\lambda+1)=0,\ \lambda=5,\ \lambda=-1$

$(-4-\lambda)(-1-\lambda)+2=0$

$4+\lambda+4\lambda+\lambda^2=0$

$\lambda^2+5\lambda+6=0$

$(\lambda+3)(\lambda+2)=0,\ \lambda=-3,\ \lambda=-2$

$\frac{2\pm\sqrt{4-4(1)(5)}}{2}=\frac{2\pm\sqrt{-16}}{2}$

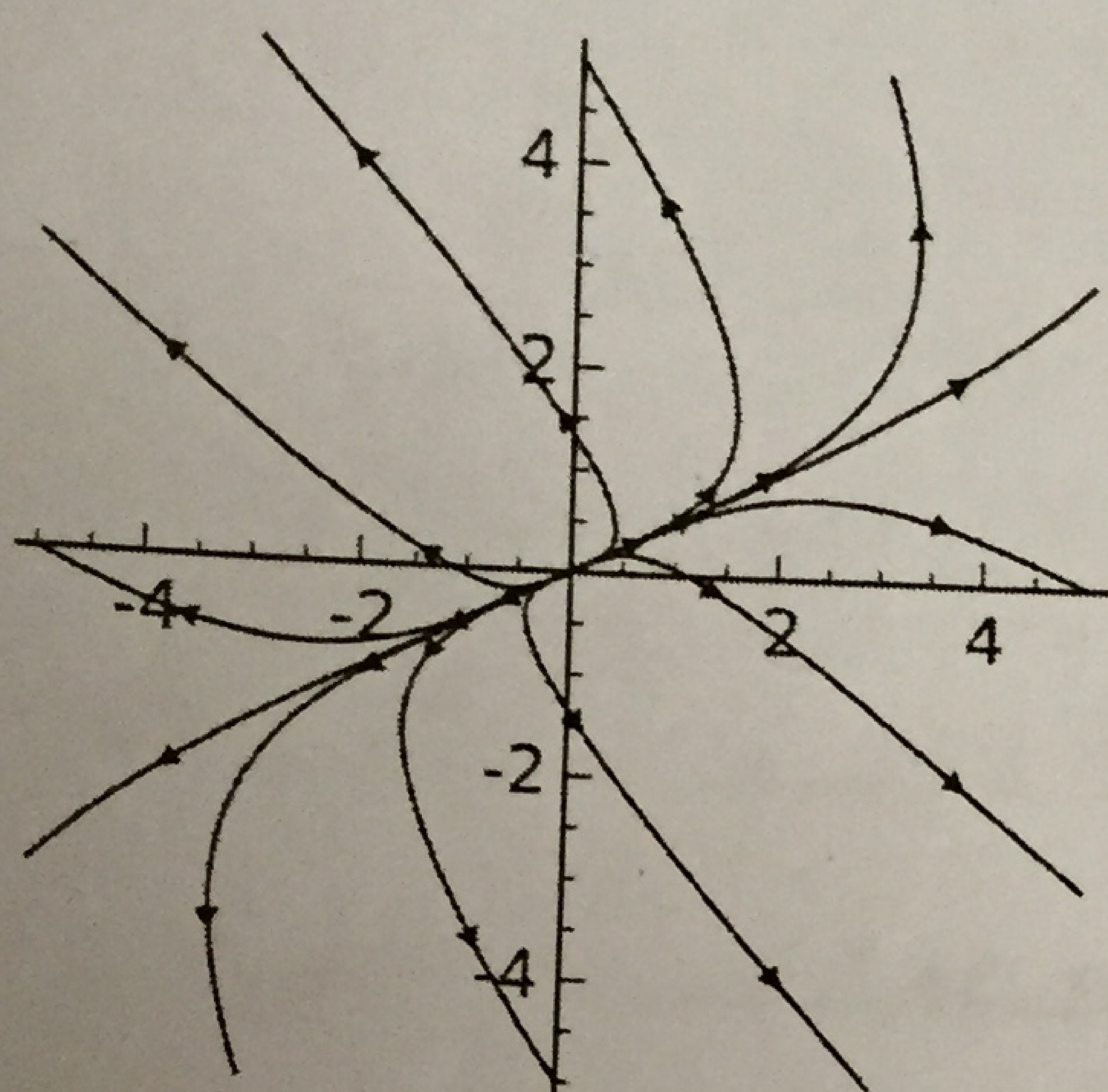
$=\frac{2\pm4i}{2}=1\pm2i$

$-\lambda(-1-\lambda)+16=0$

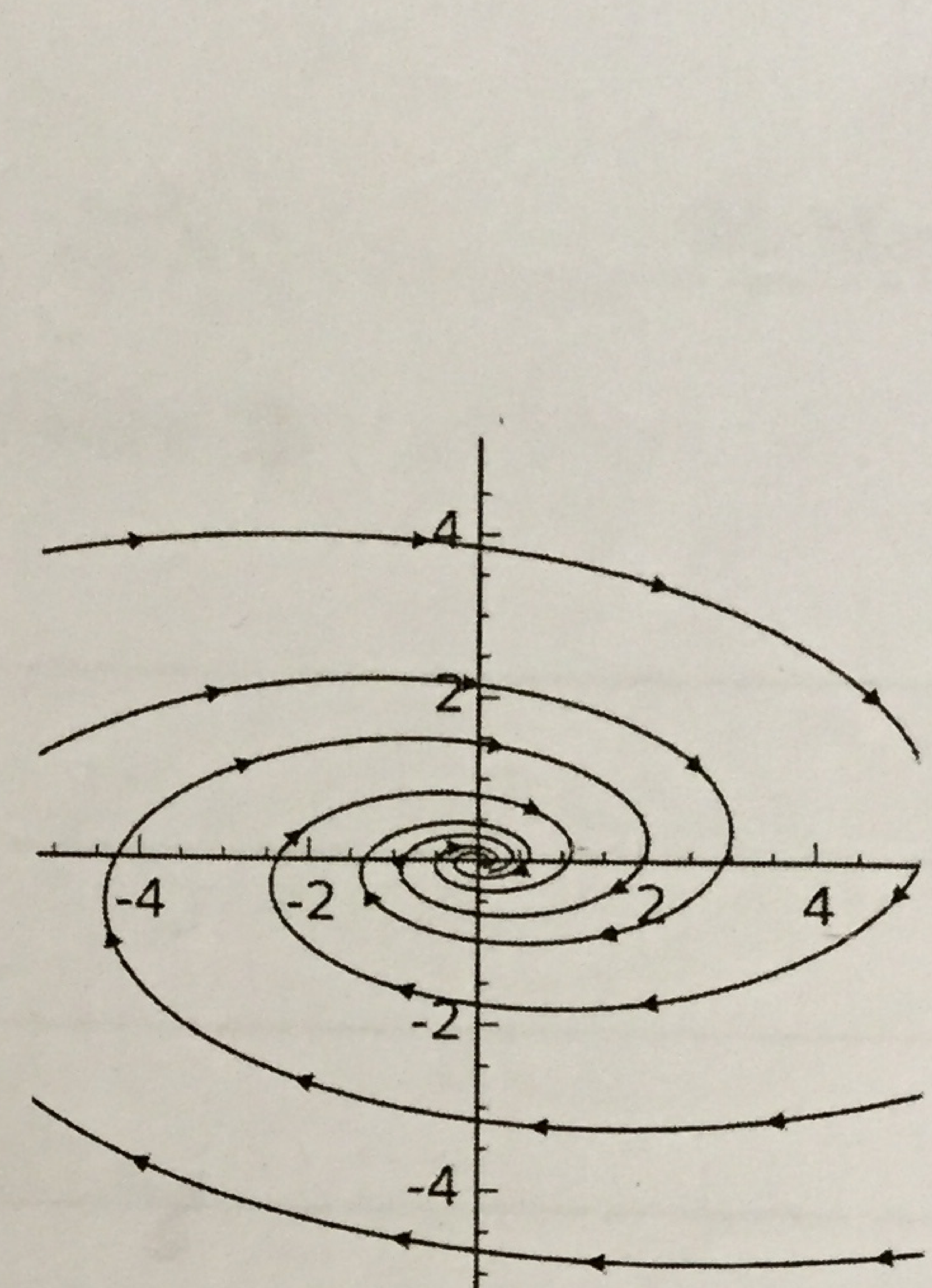
$(2-\lambda)(3-\lambda)-2=0$

$6-3\lambda-2\lambda+\lambda^2-2=0$

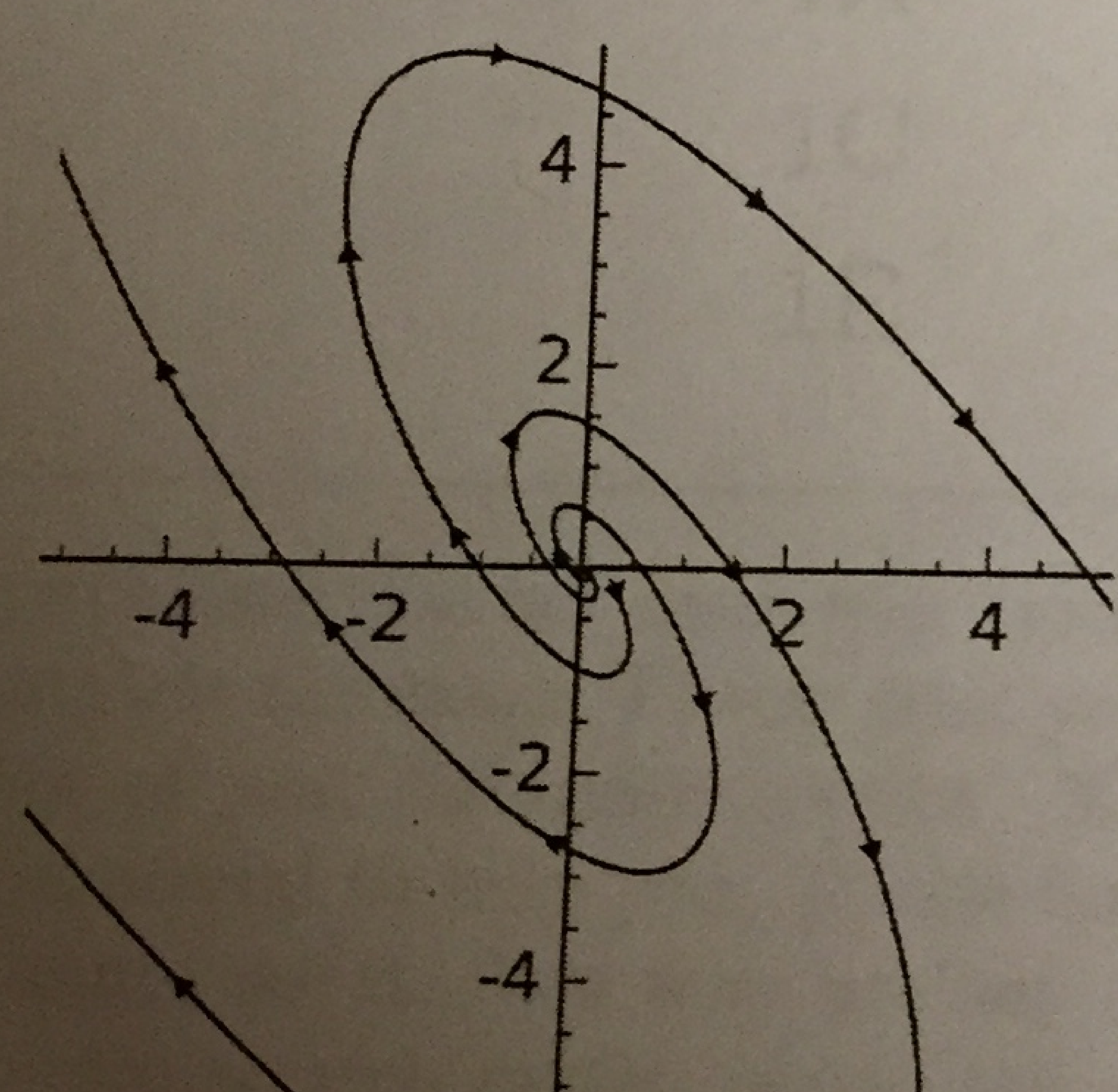
A



B



C



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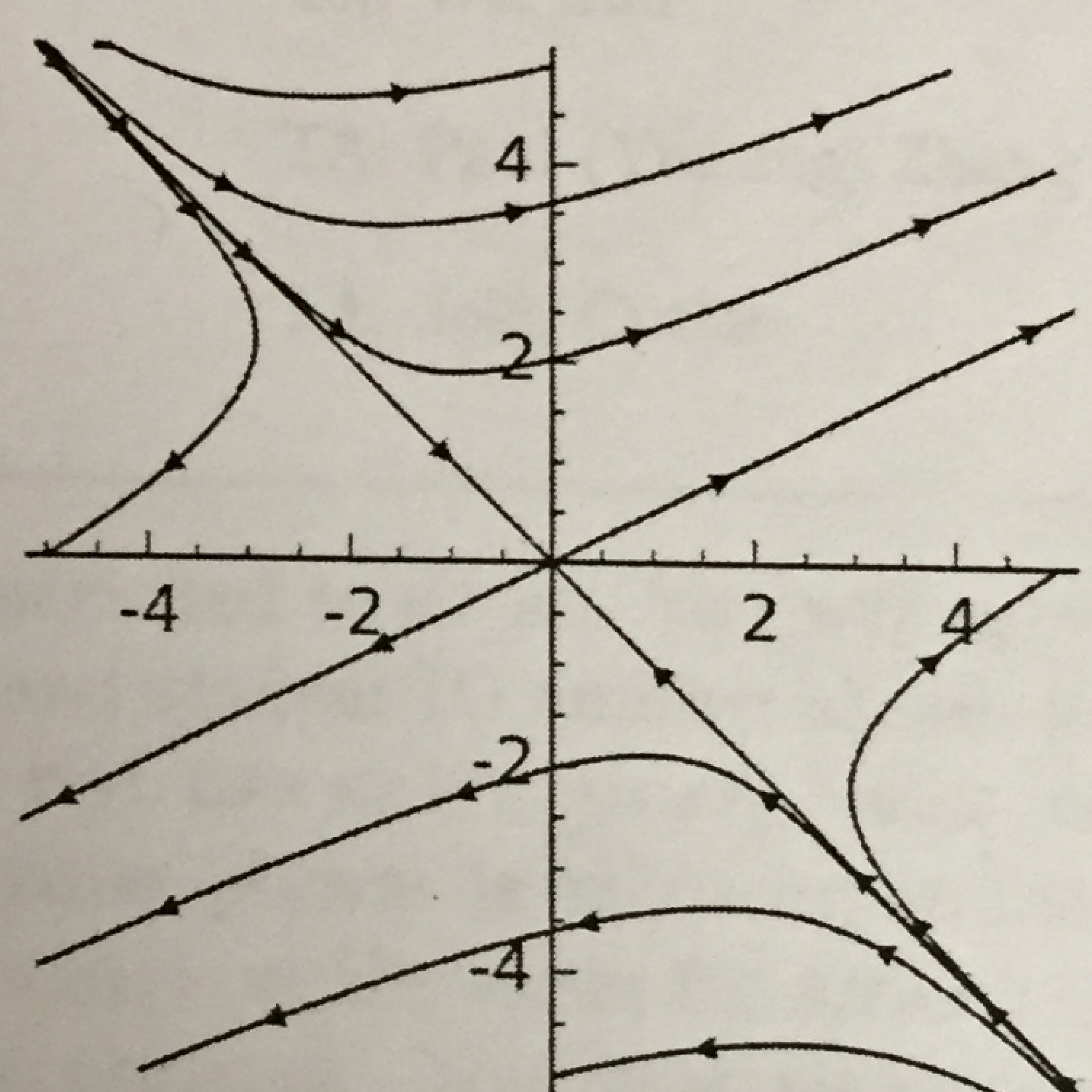

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