Math 33B MT2 Q1 · Show that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions

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Score: 100/100

· Then find a solution to the initial value problems.

$$y'' + 2y' - 3y = 0, \quad y_1(t) = e^t, \quad y_2(t) = e^{-3t}, \quad y(0) = 1, \quad y'(0) = -2$$

For $y_1(t)$ and $y_2(t)$ to form a fundamental set of solutions, $W(t) \neq 0$ for all $t \in I$ (I is the interval on which y_1 and y_2 are defined.)

The Wronskian of y_1 and y_2 is $W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$.

$y_1(t) = e^t \to y_1'(t) = e^t$

$y_2(t) = e^{-3t} \to y_2'(t) = -3e^{-3t}$

$$W(t) = (e^t)(-3e^{-3t}) - (e^{-3t})(e^t) = -3e^{-2t} - e^{-2t} = -4e^{-2t}.$$

Since $W(t)$ is an exponential function, it is never zero.

Therefore $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions.

Now solve the IVP: We know the general solution is $y(t) = c_1e^t + c_2e^{-3t} = 0$. Its derivative is $y'(t) = c_1e^t + c_2(-3e^{-3t})$.

Plug in initial conditions:

$y(0) = c_1\cancel{e^0} + c_2\cancel{e^{-3(0)}} = 1 \Rightarrow c_1 + c_2 = 1$

$y'(0) = c_1\cancel{e^0} + c_2(-3\cancel{e^{-3(0)}}) \Rightarrow c_1 - 3c_2 = -2$

$\left.\right\}$ system of eq $\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -3 & -2 \end{array}\right]$

Simplify matrix: $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -3 & -2 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -4 & -3 \end{array}\right] \xrightarrow{\div -4} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 3/4 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|c} 1 & 0 & 1/4 \\ 0 & 1 & 3/4 \end{array}\right] \rightarrow \begin{array}{l} c_1 = 1/4 \\ c_2 = 3/4 \end{array}$

Our solution to the IVP is:

$$\boxed{y(t) = \tfrac{1}{4}e^t + \tfrac{3}{4}e^{-3t}}$$

Math 33B MT2 Q2: For the following IVP's, find the solution $y(t)$, using the characteristic polynomial.

① $y''+4y'+13y=0,\quad y(0)=2,\quad y'(0)=-1$

The characteristic polynomial is $\lambda^2+4\lambda+13=0 \Rightarrow \lambda=\frac{-4\pm\sqrt{4^2-4(1)(13)}}{2}=\frac{-4\pm\sqrt{16-52}}{2}=\frac{-4\pm\sqrt{-36}}{2}=\frac{-4\pm 6i}{2}\Rightarrow \lambda=-2+3i,\ -2-3i$

The general solution is: $y(t)=c_1e^{-2t}\cos(3t)+c_2e^{-2t}\sin(3t)$

The derivative is: $y'(t)=-2c_1e^{-2t}\cos(3t)-3c_1e^{-2t}\sin(3t)-2c_2e^{-2t}\sin(3t)+3c_2e^{-2t}\cos(3t)$

Plug in values: $y(0)=c_1\cancel{e^{-2(0)}}^{1}\cancel{\cos(3\cdot 0)}^{1}+c_2\cancel{e^{-2(0)}}^{1}\cancel{\sin(3\cdot 0)}^{0}=2\Rightarrow c_1=2$

$y'(0)=-2(2)\cancel{e^{-2(0)}}^{1}\cancel{\cos(3\cdot 0)}^{1}-\cancel{3(2)e^{-2(0)}\sin(3\cdot 0)}^{0}-2c_2\cancel{e^{-2(0)}\sin(3\cdot 0)}+3c_2\cancel{e^{-2(0)}}^{1}\cancel{\cos(3\cdot 0)}^{1}=-1\Rightarrow -4+3c_2=-1\Rightarrow 3c_2=3\Rightarrow c_2=1$

The actual solution to the IVP is: $\boxed{y(t)=2e^{-2t}\cos(3t)+e^{-2t}\sin(3t)}$

② $y''-4y'-5y=0,\quad y(1)=-1,\quad y'(1)=-1$

The characteristic polynomial is $\lambda^2-4\lambda-5=0\Rightarrow(\lambda-5)(\lambda+1)=0\Rightarrow\lambda=5,\ \lambda=-1$

The general solution is: $y(t)=c_1e^{5t}+c_2e^{-t}$

The derivative is: $y'(t)=5c_1e^{5t}-c_2e^{-t}$

Plug in values: $y(1)=c_1e^{5(1)}+c_2e^{-(1)}=-1\Rightarrow e^5c_1+e^{-1}c_2=-1$

$y'(1)=5c_1e^{5(1)}-c_2e^{-(1)}=-1\Rightarrow 5e^5c_1-e^{-1}c_2=-1$

$$\left[\begin{array}{cc|c} e^5 & e^{-1} & -1\\ 5e^5 & -e^{-1} & -1\end{array}\right]\xrightarrow{-5R_1}\left[\begin{array}{cc|c} e^5 & e^{-1} & -1\\ 0 & -6e^{-1} & 4\end{array}\right]\xrightarrow{+\frac16 R_2}\left[\begin{array}{cc|c} e^5 & 0 & -1/3\\ 0 & -6e^{-1} & 4\end{array}\right]\begin{array}{l}\times e^{-5}\to\\ \times -\frac16 e\to\end{array}$$

$$\to\left[\begin{array}{cc|c} 1 & 0 & -\frac13 e^{-5}\\ 0 & 1 & -\frac23 e\end{array}\right]\qquad \begin{array}{l} c_1=-\frac13 e^{-5}\\ c_2=-\frac23 e\end{array}$$

The actual solution to the IVP is: $y(t)=-\frac13 e^{-5}e^{5t}-\frac23 e\,e^{-t}\Rightarrow\boxed{y(t)=-\frac13 e^{5t-5}-\frac23 e^{1-t}}$

Math 33B MT2 Q3. Given the equation $y''-6y'-7y=-9e^{-2t}$, $y(0)=6$, $y'(0)=3$:

① Find the general solution of the associated homogeneous equation. ($y''-6y'-7y=0$)

The characteristic polynomial is: $\lambda^2-6\lambda-7=0 \Rightarrow (\lambda-7)(\lambda+1)=0 \Rightarrow \lambda=7,\ \lambda=-1$

The general solution to the homogeneous eq. is $\boxed{y_c(t)=c_1e^{7t}+c_2e^{-t}}$

② Use the method of undetermined coefficients to find a particular solution.

The forcing term $g(t)=-9e^{-2t}$ is of the form ae^{rt}, so the trial solution is $y_p(t)=ae^{-2t}$.

Plug $y_p(t)$ into the equation: $y_p''(t)-6y_p'(t)-7y_p(t)=-9e^{-2t}$. $y_p'(t)=-2ae^{-2t}$, $y_p''(t)=4ae^{-2t}$

$\Rightarrow 4ae^{-2t}-6(-2ae^{-2t})-7ae^{-2t}=-9e^{-2t} \Rightarrow \cancel{e^{-2t}}(4a+12a-7a)=-9\cancel{e^{-2t}} \Rightarrow 9a=-9 \Rightarrow a=-1$

A particular solution to the eq. is $\boxed{y_p(t)=-e^{-2t}}$

③ Write down the general solution of the given equation.

The general solution to the given eq. is $y(t)=y_c(t)+y_p(t) \Rightarrow \boxed{y(t)=c_1e^{7t}+c_2e^{-t}-e^{-2t}}$

④ Find the solution satisfying the given initial conditions.

$y'(t)=7c_1e^{7t}-c_2e^{-t}+2e^{-2t}$

$y(0)=c_1\cancel{e^{7(0)}}^1+c_2\cancel{e^{-(0)}}^1-\cancel{e^{-2(0)}}^1=6 \Rightarrow c_1+c_2-1=6 \Rightarrow c_1+c_2=7$

$y'(0)=7c_1\cancel{e^{7(0)}}-c_2\cancel{e^{-(0)}}+2\cancel{e^{-2(0)}}=3 \Rightarrow 7c_1-c_2+2=3 \Rightarrow 7c_1-c_2=1$

$\Rightarrow \left[\begin{array}{cc|c}1&1&7\\7&-1&1\end{array}\right] \xrightarrow{-7R_1} \left[\begin{array}{cc|c}1&1&7\\0&-8&-48\end{array}\right] \xrightarrow{\times -1/8} \left[\begin{array}{cc|c}1&1&7\\0&1&6\end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|c}1&0&1\\0&1&6\end{array}\right]$

$c_1=1$

$c_2=6$

The actual solution to the IVP is:

$\boxed{y(t)=e^{7t}+6e^{-t}-e^{-2t}}$

MATH 33B MT2 Q4 · Solve the IVP ($y'' + y' - 2y = 2t$; $y(0) = 0$; $y'(0) = 1$) in three steps:

① Find the general solution of the associated homogeneous linear diff eq: $y'' + y' - 2y = 0$

The characteristic polynomial is $\lambda^2 + \lambda - 2 = 0$. $\Rightarrow (\lambda - 1)(\lambda + 2) = 0 \Rightarrow \lambda = 1, \lambda = -2$

The general solution to the homogeneous eq. is $\boxed{y_c(t) = c_1 e^t + c_2 e^{-2t}}$

② Find one particular solution to the following inhomogeneous linear eq: $y'' + y' - 2y = 2t$

The forcing term $g(t) = 2t$ is a polynomial of degree 1, so the trial solution is $y_p(t) = a_1 t + a_0$.

Plug $y_p(t)$ into the equation: $y_p''(t) + y_p'(t) - 2y_p(t) = 2t$. Here $y_p'(t) = a_1$ and $y_p''(t) = 0$.

$\Rightarrow 0 + a_1 - 2(a_1 t + a_0) = 2t \Rightarrow a_1 - 2a_1 t - 2a_0 = 2t \Rightarrow \underbrace{-2a_1 t}_{a} + \underbrace{a_1 - 2a_0}_{b} = \underset{a}{2t} + \underset{b}{0}$

$-2a_1 = 2 \rightarrow a_1 = -1$

$a_1 - 2a_0 = 0 \rightarrow -1 - 2a_0 = 0 \rightarrow -2a_0 = 1 \rightarrow a_0 = -\frac{1}{2}$

A particular solution to the eq. is $\boxed{y_p(t) = -t - \frac{1}{2}}$

③ Use the answers from Steps ① and ② to find the solution to the original IVP. ($y'' + y' - 2y = 2t$; $y(0) = 0$; $y'(0) = 1$)

The general solution to the diff. eq. is $y(t) = y_c(t) + y_p(t) \Rightarrow y(t) = c_1 e^t + c_2 e^{-2t} - t - \frac{1}{2}$

The derivative is: $y'(t) = c_1 e^t - 2c_2 e^{-2t} - 1$

Plug in values: $y(0) = c_1 \cancelto{1}{e^0} + c_2 \cancelto{1}{e^{-2(0)}} - 0 - \frac{1}{2} = 0 \Rightarrow c_1 + c_2 - \frac{1}{2} = 0 \Rightarrow c_1 + c_2 = \frac{1}{2}$

$y'(0) = c_1 \cancelto{1}{e^0} - 2c_2 \cancelto{1}{e^{-2(0)}} - 1 = 1 \Rightarrow c_1 - 2c_2 - 1 = 1 \Rightarrow c_1 - 2c_2 = 2$

Solve system of eqs: $\left[\begin{array}{cc|c} 1 & 1 & \frac{1}{2} \\ 1 & -2 & 2 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{cc|c} 1 & 1 & \frac{1}{2} \\ 0 & -3 & \frac{3}{2} \end{array}\right] \xrightarrow{\times -\frac{1}{3}} \left[\begin{array}{cc|c} 1 & 1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \end{array}\right]$ $\quad c_1 = 1$, $c_2 = -\frac{1}{2}$

The actual solution to the IVP is: $\boxed{y(t) = e^t - \frac{1}{2}e^{-2t} - t - \frac{1}{2}}$