MATH 33B: MIDTERM 2

Question 1.

For the following differential equation, show that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions for the given differential equation (check linear independent by computing Wronskian or by definition of linear independent). Then find a solution to the initial value problems:
$y'' + 2y' - 3y = 0, y_1(t) = e^t, y_2(t) = e^{-3t}, y(0) = 1, y'(0) = -2$

Understand the definition of a fundamental set of solutions.
Check linear independent by computing Wronskian

$$W(t) = \det \begin{bmatrix} e^t & e^{-3t} \\ (e^t)' & (e^{-3t})' \end{bmatrix} = \det \begin{bmatrix} e^t & e^{-3t} \\ e^t & -3e^{-3t} \end{bmatrix}$$

$$= -3e^{-2t} - e^{-2t}$$

$$= -4e^{-2t} \quad \text{never zero}$$

(1) It is easy to check that e^t and e^{-3t} are solutions to the given differential equation, so we just focus on showing they are linearly independent. Suppose we have A, B real constants with

$$Ae^t + Be^{-3t} = 0$$

Plugging in $t = 0$ gives

$$A + B = 0$$

On the other hand, if we differentiate each side we get

$$Ae^t - 3Be^{-3t} = 0$$

Plugging in $t = 0$ for this gives

$$A - 3B = 0$$

We can solve and see that $A = 3B = -B$, so $B = A = 0$. Thus the two given solutions are linearly independent, and since the differential equation is second order, they therefore form a fundamental set of solutions. (Alternatively, one could compute the Wronskian and show that it is nonzero).

A general solution is then of the form $y(t) = C_1e^t + C_2e^{-3t}$. The initial conditions give us

$$y(0) = 1 = C_1 + C_2$$

$$y'(0) = -2 = C_1 - 3C_2$$

Solving gives $C_1 = 1 - C_2 = -2 + 3C_2$ so that $C_2 = \frac{3}{4}$ and $C_1 = \frac{1}{4}$. This gives a solution to the given initial value problem:

$$y(t) = \frac{1}{4}e^t + \frac{3}{4}e^{-3t}$$

Question 2.

For the following initial value problems (use characteristic polynomial) , find the solution $y(t)$.

(1) $y'' + 4y' + 13y = 0, \quad y(0) = 2, \quad y'(0) = -1$

(2) $y'' - 4y' - 5y = 0, y(1) = -1, y'(1) = -1$

(1) The characteristic polynomial is $\lambda^2+4\lambda+13$, which has roots $\lambda = -2\pm 3i$. Thus the general solution is $y(t) = e^{-2t}(A\cos(3t)+B\sin(3t))$. The initial conditions give the system $\begin{cases} 2 = A \\ -1 = -2A + 3B \end{cases}$ This has solution $A = 2, B = 1$. Thus $y(t) = e^{-2t}(2\cos(3t) + \sin(3t)$

(2) The characteristic polynomial is

$$\chi(\lambda) = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1),$$

so the general solution has the form

$$y(t) = Ae^{5t} + Be^{-t}.$$

The initial conditions $y(1) = -1$ and $y'(1) = -1$ impose the constraints

$$Ae^5 + Be^{-1} = -1 \quad \text{and} \quad 5Ae^5 - Be^{-1} = -1.$$

Hence

$$A = -\frac{1}{3}e^{-5} \quad \text{and} \quad B = -\frac{2}{3}e,$$

and the solution to the initial value problem is

$$y(t) = -\frac{1}{3}e^{5(t-1)} - \frac{2}{3}e^{1-t}.$$

Question 3.

Perform the following tasks for the equations below:

(1) Find a general solution of the associated homogeneous equation.

(2) Use the method of undetermined coefficients to find a particular solution of the given equation.

(3) Write down the general solution of the given equation.

(4) Find the solution satisfying the given initial conditions.

$$y'' - 6y' - 7y = -9e^{-2t}, \quad y(0) = 6, \quad y'(0) = 3$$

Solution:
The characteristic polynomial is $\lambda^2 - 6\lambda - 7$, which has roots $\lambda = 7, -1$. Thus the general homogeneous solution is $y_h(t) = c_1 e^{7t} + c_2 e^{-t}$.

For a particular solution, we will guess $y_p(t) = Ae^{-2t}$. Plugging y_p into the inhomogeneous equation, we obtain the equation $9Ae^{-2t} = -9e^{-2t}$, hence $A = -1$. Thus $y_p(t) = -e^{-2t}$ is a particular solution.

The general inhomogeneous solution is given by $y = y_p + y_h$, thus $y(t) = -e^{-2t} + c_1 e^{7t} + c_2 e^{-t}$.

Plugging in the initial conditions, we obtain equations

$$\begin{aligned} 6 &= y(0) = -1 + c_1 + c_2 \\ 3 &= y'(0) = 2 + 7c_1 - c_2 \end{aligned}$$

In standard form, this becomes

$$\begin{aligned} 7 &= c_1 + c_2 \\ 1 &= 7c_1 - c_2 \end{aligned}$$

Solving, we have $c_1 = 1, c_2 = 6$. Thus the final solution is $y(t) = -e^{-2t} + e^{7t} + 6e^{-t}$.

Question 4.

In this question you will solve the following initial value problem in three steps:

(i) $y'' + y' - 2y = 2t$

(ii) $y(0) = 0, y'(0) = 1$

Step 1: Find the general solution of the associated homogeneous linear differential equation

$y'' + y' - 2y = 0.$

Step 2: Find one particular solution to the following inhomogeneous linear differential equation:

$y'' + y' - 2y = 2t.$

Hint: Use the method of undetermined coefficients to guess an appropriate trial solution $y_p(t)$.

Step 3: Use the answers you got in Steps 1 and 2 to find he solution to the original initial value problem:

(i) $y'' + y' - 2y = 2t$

(ii) $y(0) = 0, y'(0) = 1$

Solution:
Step 1: The characteristic polynomial is $\lambda^2 + \lambda - 2$, which has roots $\lambda = -2, 1$. Thus the general homogeneous solution is $y_h(t) = c_1 e^{-2t} + c_2 e^t$.

Step 2: We will guess $y_p(t) = At + B$. Plugging this guess into the inhomogeneous equation, we obtain the equation $-2At + (A - 2B) = 2t$. It follows that $-2A = 2$ and $A - 2B = 0$. Solving, we obtain $A = -1, B = -\frac{1}{2}$. Thus a particular solution is $y_p(t) = -t - \frac{1}{2}$.

Step 3: Our general inhomogeneous solution is $y(t) = y_p(t) + y_h(t) = -t - \frac{1}{2} + c_1 e^{-2t} + c_2 e^t$. Plugging in the initial condition, we obtain:

$$\begin{aligned} 0 &= y(0) = -\frac{1}{2} + c_1 + c_1 \\ 1 &= y'(0) = -1 - 2c_1 + c_2 \end{aligned}$$

Solving, we obtain $c_1 = -\frac{1}{2}, c_2 = 1$. Thus our solution is $y(t) = -t - \frac{1}{2} - \frac{1}{2}e^{-2t} + e^t$