Q1
10 Points

[New instruction: on *all* of these problems (Questions 1-5) you may use a calculator/compute for *arithmetic only*, i.e., simplifying expression which involve $+, \cdot, -, /$.]

Answer the following multiple-choice questions. Double-check your answers, since no partial score would be given for these questions.

Q1.1
2 Points

How many solutions does the following system have:

$$\begin{aligned} x + 2y + 3z &= 1 \\ x + 3y + 4z &= 3 \\ x + 4y + 5z &= 4 \end{aligned}$$

- ○ one
- ○ infinitely many
- ◉ none

Q1.2
2 Points

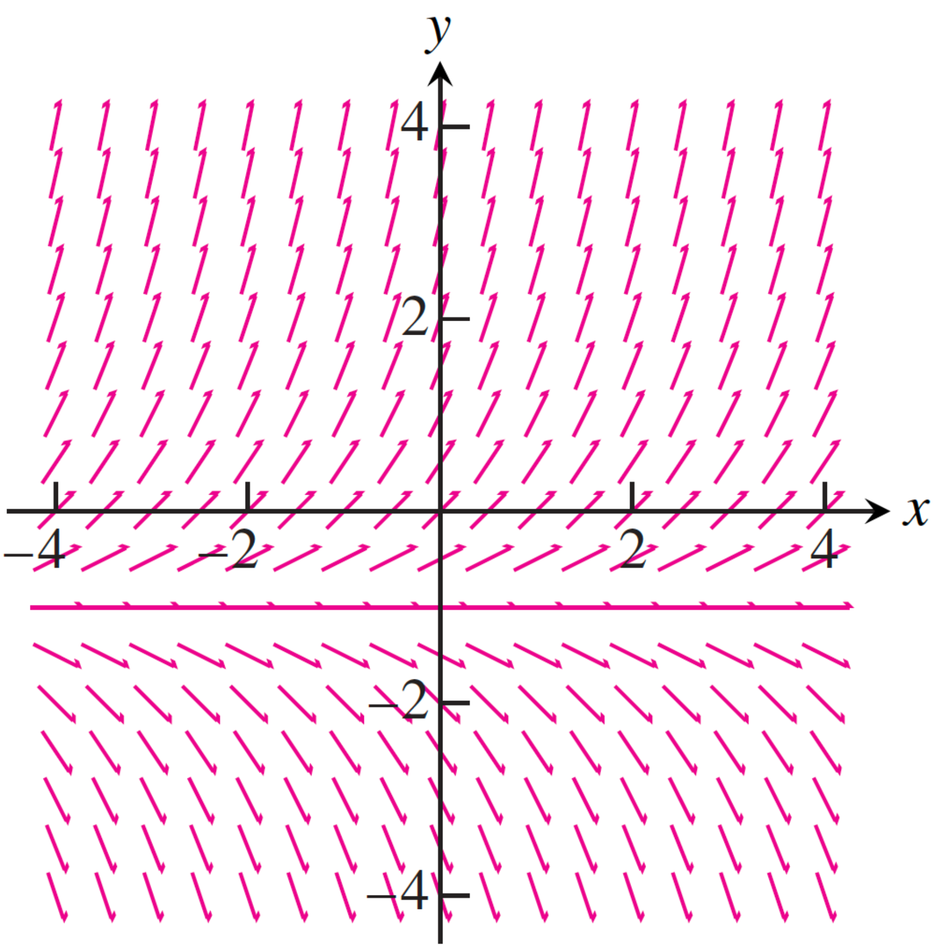
Which of the following differential equations are linear?

- [ ] $y' + ty^2 = \sqrt{t}$
- [x] $t - y' = \frac{1}{\sin(t)} y$
- [ ] $y' = \frac{1}{t} - \frac{1}{y}$
- [x] $y \sin(t) = t^2 y' - e^t$

Q1.3

2 Points

Choose the differential equation whose direction field is graphed below



- ○ $y' = x + y$
- ◉ $y' = y + 1$
- ○ $y' = y^2 + 1$

Q1.4

2 Points

Which of the following equations can be rearranged into separable equations?

- [ ] $y' - y^2 = t$
- [x] $y' + yt^2 = t^2$
- [x] $ty' - e^t \sin(y) = e^t$
- [ ] $y' - \sin(t)y = t$

Q1.5

2 Points

Which of the following differential forms are exact?

- [x] $(y^2 - t^2)\,dt + (2ty - y)\,dy$
- [ ] $ye^t\,dt - e^t\,dy$
- [x] $3\cos(3t - y)\,dt - \cos(3t - y)\,dy$
- [ ] $(2t^2y - 1)\,dt + t^3\,dy$

Q2

20 Points

Solve the following initial value problem:
$(1+t)y'(t) + y(t) = 0, y(1) = 1.$

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2) $(1+t)y'(t) + y(t) = 0 \quad y(1) = 1$

$y'(t) + \frac{1}{1+t} y(t) = 0 \quad \mu(t) = e^{\int \frac{1}{1+t} dt} = e^{\ln|1+t|} = 1+t$

$(1+t)y' + y = 0$

$(1+t)y = c$

$y = \frac{c}{1+t} \quad y(1) = 1 = \frac{c}{2} \quad c = 2 \quad \boxed{y = \frac{2}{1+t}}$

Q3

20 Points

Solve the following initial value problem:

$$y' = \frac{t^2+4}{3t^3+4t^2-4t} \quad y(1) = 1.$$

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3) $\frac{dy}{dt} = \frac{t^2+4}{3t^3+4t^2-4t} \quad y(1) = 1$

$dy = \frac{t^2+4}{3t^3+4t^2-4t} dt$

$y = \int \frac{t^2+4}{3t^3+4t^2-4t} dt \rightarrow \frac{A}{t} + \frac{B}{t+2} + \frac{C}{3t-2}$

$t=0: \; 4 = -4A \quad A = -1$
$t=-2: \; 8 = 16B \quad B = 1/2$
$t=\frac{2}{3}: \; \frac{40}{9} = \frac{16}{9}C \quad C = \frac{5}{2}$

$y = \int \left[\frac{-1}{t} + \frac{1}{2(t+2)} + \frac{5}{2(3t-2)} \right] dt$

$y = -\ln|t| + \frac{1}{2}\ln|t+2| + \frac{5}{6}\ln|3t-2| + C \quad u = 3t-2, \; du = 3\,dt$

$y(1) = 1 = \frac{1}{2}\ln(3) + C \quad C = 1 - \frac{1}{2}\ln(3)$

$y = -\ln|t| + \frac{1}{2}\ln|t+2| + \frac{5}{6}\ln|3t-2| + 1 - \frac{1}{2}\ln 3$ ← can take out abs value because IOE: $(0, \infty)$

$\Rightarrow \boxed{y = -\ln(t) + \frac{1}{2}\ln(t+2) + \frac{5}{6}\ln(3t-2) + 1 - \frac{1}{2}\ln 3}$

Q4

25 Points

Consider the following initial value problem

$y' = (y+2)(y-4), y(0) = 3$

(1) Find the solution $y(t)$ in an explicit form.

(2) Use the theory of limits to find the behavior of $y(t)$ as $t \to -\infty$ and as $t \to +\infty$.

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4.1) $y' = (y+2)(y-4) \qquad y(0) = 3$

$\frac{dy}{dt} = (y+2)(y-4)$

$\int \frac{dy}{(y+2)(y-4)} = \int dt$

$-\frac{1}{6}\int \frac{1}{y+2}dy + \frac{1}{6}\int \frac{1}{y-4}dy = t + C$

$-\frac{1}{6}\ln|y+2| + \frac{1}{6}\ln|y-4| = t + C$

$\frac{1}{6}\left(\ln|y-4| - \ln|y+2|\right) = t + C$

$\frac{1}{6}\ln\left|\frac{y-4}{y+2}\right| = t + C$

$\ln\left|\frac{y-4}{y+2}\right| = 6t + 6C$

$e^{6t+6C} = \frac{y-4}{y+2}$

$De^{6t} = \frac{y-4}{y+2} \qquad y(0)=3 \rightarrow D = \frac{-1}{5}$

$-\frac{1}{5}e^{6t} = \frac{y-4}{y+2}$

$y - 4 = -\frac{1}{5}e^{6t}y - \frac{2}{5}e^{6t}$

$5y - 20 = -e^{6t}y - 2e^{6t}$

$5y + e^{6t}y = -2e^{6t} + 20$

$y(5 + e^{6t}) = -2e^{6t} + 20$

$\boxed{y = \frac{-2e^{6t} + 20}{5 + e^{6t}}}$

4.2)

$\lim_{t \to -\infty} \frac{-2e^{6t} + 20}{5 + e^{6t}}$

$= \frac{-2e^{6(-\infty)} + 20}{5 + e^{6(-\infty)}}$

$= \frac{-2\frac{1}{e^{6\infty}} + 20}{5 + \frac{1}{e^{6\infty}}}$

$= \frac{-2(0) + 20}{5 + (0)} = \frac{20}{5} = \boxed{4}$

$\lim_{t \to +\infty} \frac{-2e^{6t} + 20}{5 + e^{6t}}$

$= \frac{-2e^{6\infty} + 20}{5 + e^{6\infty}} = \frac{-\infty}{\infty}$ ← indeterminate

$\lim_{t \to +\infty} \frac{-12e^{6t}}{6e^{6t}} = \boxed{-2}$

Q5

25 Points

Solve the following initial value problem and determine the respective interval of existence:

$y' = e^{t-2y}, y(0) = 0.$

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5) $y' = e^{t-2y} \quad y(0) = 0$

$$y' = e^t e^{-2y}$$

$$\frac{dy}{e^{-2y}} = e^t \, dt$$

$$\int e^{2y} dy = \int e^t dt$$

$$\frac{1}{2}e^{2y} = e^t + C$$

$$e^{2y} = 2e^t + C$$

$$2y = \ln(2e^t + C)$$

$$y = \frac{1}{2}\ln(2e^t + C)$$

$$y(0) = 0 = \frac{1}{2}\ln(2 + C) \quad C = -1$$

$$\boxed{y = \frac{1}{2}\ln(2e^t - 1)} \quad \boxed{\text{IOE}: (\ln\tfrac{1}{2}, \infty)}$$

$$e^t > \frac{1}{2} \quad t > \ln\frac{1}{2}$$

Midterm1

GRADED

STUDENT

KIM, JIIN

TOTAL POINTS

100 / 100 pts

QUESTION 1

(no title) **10** / 10 pts

1.1	(no title)	**2** / 2 pts
1.2	(no title)	**2** / 2 pts
1.3	(no title)	**2** / 2 pts
1.4	(no title)	**2** / 2 pts
1.5	(no title)	**2** / 2 pts

QUESTION 2

(no title) **20** / 20 pts

QUESTION 3

(no title) **20** / 20 pts

QUESTION 4

(no title) **25** / 25 pts

QUESTION 5

(no title) **25** / 25 pts