University of California, Los Angeles
Fall 2021

Instructor: C. Wang
Date: Oct. 20, 2021

MATH 33B: Differential Equations
Midterm Exam 1

Last Name Shen

First Name Yu

Student ID 505572058

TA Name Thomas Brown Dis 2A

Last Name Shen, First Name Yu,
Student ID 50557058

Q1

25 Points

Check whether the following differential form is closed and exact.

$(2t + 3y)dt + (3t - 6y)dy$

$\frac{\partial}{\partial y}(2t+3y) = 3$

$\frac{\partial}{\partial t}(3t-6y) = 3$

So the form is closed.

Since $2t+3y$ and $3t-6y$ are continuously differentiable, and have a rectangle domain $\mathbb{R} \times \mathbb{R}$, the form is also exact.

Last Name Shen, First Name Yu,
Student ID 50557058

Q2

25 Points

Solving the following initial value problem (no need to give the interval of existence):

$y' + (2/t)y = sin(t)/t^2, y(\pi/2) = 2/\pi$

$$\mu(t) = e^{\int_{\frac{\pi}{2}}^{t} \frac{2}{s} ds} = e^{[2\ln|s|]_{\frac{\pi}{2}}^{t}} = e^{2\ln(t) - 2\ln(\frac{\pi}{2})}$$

$$= e^{(\ln(t) - \ln(\frac{\pi}{2}))^2}$$

$$= e^{\ln(\frac{2t}{\pi})^2}$$

$$= \frac{4t^2}{\pi^2}$$

$$y(t) = \frac{1}{\mu(t)} \int_{\frac{\pi}{2}}^{t} \mu(s) g(s)\, ds + \frac{\frac{2}{\pi}}{\mu(t)}$$

$$= \frac{\pi^2}{4t^2} \int_{\frac{\pi}{2}}^{t} \frac{4s^2}{\pi^2} \cdot \frac{\sin(s)}{s^2} ds + \frac{\pi^2}{4t^2} \cdot \frac{2}{\pi}$$

$$= \frac{1}{t^2} \int_{\frac{\pi}{2}}^{t} \sin(s)\, ds + \frac{2\pi}{4t^2}$$

$$= \frac{1}{t^2} [-\cos(s)]_{\frac{\pi}{2}}^{t} + \frac{2\pi}{4t^2}$$

$$= -\frac{1}{t^2}\cos(t) + \frac{2\pi}{4t^2}$$

Last Name Shen, First Name Yu,
Student ID 50557058

Q3

25 Points

Solving the following separable differential equation (only need to give general solution in implicit form):

$dy/dt = ty$

$$\frac{dy}{dt} = ty$$

$$dy = ty\,dt$$

$$-ty\,dt + dy = 0$$

$$-t\,dt + \frac{1}{y}dy = 0$$

$$F(t,y) = \int -t\,dt + \int \frac{1}{y}dy$$

$$= -\frac{t^2}{2} + \ln|y|$$

So the general solution:

$$-\frac{t^2}{2} + \ln|y| = C$$

Last Name Shen, First Name Yu,
Student ID 50557058

Q4

25 Points

A tank contains 100 gallons of brine made by dissolving 80 lb of salt in water. Pure water runs into the tank at the rate of 4 gallons/minute, and the mixture, which is kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time t. Find the concentration of salt in the tank at any time t.

Let $Y(t)$ = amount of salt at time t, in lb

$Y'(t)$ = rate of change of amount of salt at time t, in lb/minute

= rate in − rate out

$= 0 - 4 \cdot \frac{Y}{100}$

$= -\frac{Y}{25}$

$Y'(t) = -\frac{Y}{25}$, $Y(0) = 80$

$Y' + \frac{1}{25}Y = 0$

$\mu(t) = e^{\int_0^t \frac{1}{25} ds} = e^{\left[\frac{s}{25}\right]_0^t} = e^{\frac{t}{25}}$

$Y(t) = \frac{1}{\mu(t)} \int_0^t 0 \, ds + \frac{80}{\mu(t)} = \frac{80}{\mu(t)} = 80 e^{-\frac{t}{25}}$ lb

Let $c(t)$ = concentration of salt at time t, in lb/gallon

$c(t) = \frac{Y(t)}{100} = \frac{4}{5} e^{-\frac{t}{25}}$ lb/gallon