SOLUTIONS TO MIDTERM 1

Solution 1. 1. Simplify the matrix of system:
$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 3 & 4 & 3 \\ 1 & 4 & 5 & 4 \end{array}\right) \xrightarrow{R_2-R_1\to R_1, R_3-R_1\to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 3 \end{array}\right) \xrightarrow{R_3-2R_2\to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right)$. The last row shows that the system has no solution.

2. By the general procedure of partial decomposition, the third one is the choice. Computing the decomposition results in

$$\frac{-3/2}{t+1} + \frac{-1/2}{(t+1)^2} + \frac{3/2t+0}{t^2+1},$$

so the forth option was also counted as a correct choice.

3. For the first one, the derivative depends only on y and there should be flat line at only $y = 2$. So third one is the choice. For the second one, the derivative depends only on y and should have flat lines at $y = \pm 2$. So the second one is the choice. For the last equation, the only choice left is the first one.

4. The second one can be rewritten as $y' = t^2(1-y)$. The third one can be rewritten as $y' = \frac{e^t}{t}(1+sin(y))$. They are the only right choices.

5.

$$\frac{\partial}{\partial y}(y^2 - t^2) = 2y = \frac{\partial}{\partial t}(2ty - y),$$

$$\frac{\partial}{\partial y}(ye^t) = e^t \neq -e^t = \frac{\partial}{\partial t}(-e^t),$$

$$\frac{\partial}{\partial y}(3\cos(3t-y)) = 3\sin(3t-y) = 3\sin(3t-y) = \frac{\partial}{\partial t}(-\cos(3t-y)),$$

$$\frac{\partial}{\partial y}(2t^2y - 1) = 2t^2 \neq 3t^2 = \frac{\partial}{\partial t}(t^3).$$

Thus the first one and the third one are correct.

2. *Find the general solution for the following differential equations:*

(1) $y' + ay = t^n e^{-at}$, where $a \in \mathbb{R}$ and $n \in \mathbb{N}$;

(1) We want to solve the differential equation

$$y' + ay = t^n e^{-at}$$

where $a \in \mathbb{R}$ and $n \in \mathbb{N}$. From above we already saw that e^{at} is an integrating factor. Multiplying through gives

$$(e^{at}y)' = t^n$$

Integrating gives

$$e^{at}y = \frac{t^{n+1}}{n+1} + C$$

Our final solution is then

$$y = \frac{t^{n+1}}{n+1}e^{-at} + Ce^{-at}$$

3 . *A tank contains 100 gallons of brine made by dissolving 80 lb of salt in water. Pure water runs into the tank at the rate of 4 gallons/minute, and the mixture, which is kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time t. Find the concentration of salt in the tank at any time t.*

Solution. Let $s(t)$ be the amount of salt (in pounds) in the tank. The balance equation gives

$$s'(t) = -\frac{4s(t)}{100}.$$

This equation has general solution $s(t) = ce^{-1/25t}$. The tank starts with 80lbs of salt in it, so $y(0) = 80$. Plugging this into the general solution yields

$$80 = s(0) = ce^0 = c.$$

Thus

$$s(t) = 80e^{-t/25}.$$

4. *Solve the following initial value problems and determine the respective intervals of existence.*

(1) $te^{t^2} + yy' = 0, \quad y(0) = 1;$

(1) This is also separable:

$$y\,dy = -te^{t^2}\,dt$$

which we integrate (using the u-sub $u = t^2$ on the right hand side).

$$\frac{y^2}{2} = -\frac{e^{t^2} + C}{2}$$

Plugging in the initial conditions, we get $C = -2$.

$$\Rightarrow y(t) = \sqrt{2 - e^{t^2}}$$

Note that we take the positive square root, since the initial value for y is positive. The argument of the square root cannot be negative, so $t^2 \leq \ln(2)$, or $-\sqrt{\ln(2)} \leq t \leq \sqrt{\ln(2)}$, so the interval of existence is $(-\sqrt{\ln(2)}, \sqrt{\ln(2)})$.

5. *Check that the following differential forms are exact and find the solutions to the corresponding initial value problems.*

(1) $\frac{y}{t+1}\,dt + (\ln(t+1) + 3y^2)\,dy = 0, \quad y(0) = 1;$

(1) Checking that a differential form is exact (for (t, y) belonging to a rectangle) corre-sponds to checking that the differential form is closed. We have

$$\frac{\partial}{\partial y}\frac{y}{t+1} = \frac{1}{t+1} \quad \text{and} \quad \frac{\partial}{\partial t}(\ln(t+1) + 3y^2) = \frac{1}{t+1},$$

so the differential form is exact. The general solution to the differential form will be

$$F(t, y) = C,$$

where $F(t, y)$ satisfies the following two conditions:

$$\frac{\partial}{\partial t}F(t, y) = \frac{y}{t+1} \quad \text{and} \quad \frac{\partial}{\partial y}F(t, y) = \ln(t+1) + 3y^2.$$

The first condition yields

$$F(t, y) = y\ln(t+1) + \phi(y),$$

for some (continuously differentiable) function $\phi(y)$ only depending on y. The second condition then implies that

$$\ln(t+1) + \phi'(y) = \ln(t+1) + 3y^2,$$

in other words

$$\phi(y) = y^3 + \text{constant}.$$

Hence the general solution is

$$y\ln(t+1) + y^3 = C.$$

Using the initial condition $y(0) = 1$ gives $C = 1$, so the solution to the initial value problem (in implicit form) is

$$y\ln(t+1) + y^3 = 1.$$