Catherine Hu

Math 33B Final

Problem 1

$y' + ay = t^n e^{-at}$ $\quad a \in \mathbb{R}$ and $n \in \mathbb{R}$

$\mu(t)$ integrating factor $= e^{\int a\,dt} = e^{at}$

multiply both sides

$e^{at}(y' + ay) = t^n$

$e^{at}y' + ae^{at}y = t^n$

$(ye^{at})' = t^n$

$ye^{at} = \dfrac{t^{n+1}}{n+1} + C$

$$\boxed{y = \frac{t^{n+1}}{(e^{at})(n+1)} + \frac{c}{e^{at}}}$$

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problem 2

$y''' - 4y'' - 7y' + 10y = 0$

a) let $x_1 = y$, $x_2 = y'$, $x_3 = y''$

$x_1' = x_2$, $x_2' = x_3$, $x_3' = 4x_3 + 7x_2 - 10x_1$

$$x' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & 7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

find eigenvalues of this matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & 7 & 4 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -10 & 7 & 4-\lambda \end{bmatrix}$$

$= a_{11}(\det A_{11}) - a_{21}(\det A_{21}) + a_{31}(\det A_{31})$

$= -\lambda[(-\lambda)(4-\lambda) - 7] - 0 - 10(1)$

$= -\lambda(\lambda^2 - 4\lambda - 7) - 10$

$= -\lambda^3 + 4\lambda^2 + 7\lambda - 10 = p(\lambda)$ characteristic polynomial

$-\lambda^3 + 4\lambda^2 + 7\lambda - 10 = 0$

$\lambda^3 - 4\lambda^2 - 7\lambda + 10 = 0$

$(\lambda - 1)(\lambda + 2)(\lambda - 5) = 0$

so $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 5$

b) for $\lambda_1 = 1$: find an eigenvector v_1

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -10 & 7 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -10 & 7 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_1 = x_3 \\ x_2 = x_3 \\ x_3 = \text{free} \end{array} \qquad v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

for $\lambda_2 = -2$: find an eigenvector v_2

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -10 & 7 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ -10 & 7 & 6 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_1 = \frac{1}{4} x_3 \\ x_2 = -\frac{1}{2} x_3 \\ x_3 = \text{free} \\ \hookrightarrow \text{let } x_3 = 4 \end{array} \qquad v_2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

for $\lambda_3 = 5$: find an eigenvector v_3

$$\begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ -10 & 7 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} -5 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ -10 & 7 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{25} & 0 \\ 0 & 1 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_1 = \frac{1}{25} x_3 \\ x_2 = \frac{1}{5} x_3 \\ x_3 = \text{free} \\ \hookrightarrow \text{let } x_3 = 25 \end{array} \qquad v_3 = \begin{bmatrix} 1 \\ 5 \\ 25 \end{bmatrix}$$

$$\vec{x} = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 + C_3 e^{\lambda_3 t} v_3$$

$$\vec{x} = C_1 e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + C_3 e^{5t} \begin{bmatrix} 1 \\ 5 \\ 25 \end{bmatrix}$$

c) therefore :

$$\boxed{y = C_1 e^{t} + C_2 e^{-2t} + C_3 e^{5t}}$$

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Problem 3

$\vec{x}' = A\vec{x}$ where $A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{bmatrix} = (1-\lambda)(-3-\lambda) + 5 = \lambda^2 + 2\lambda + 2$

$p(\lambda) = \lambda^2 + 2\lambda + 2 = 0$

$\frac{-2 \pm \sqrt{4-4(2)}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

$\lambda_1 = -1+i$, $\lambda_2 = -1-i$

for $\lambda_1 = -1+i$, find eigenvector v_1:

$(A - \lambda I)\vec{v} = 0$

$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right]$

$v_1 = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$

for $\lambda_2 = -1-i$, find eigenvector v_2:

$(A - \lambda I)\vec{v} = 0$

$\begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 1 & -2+i & 0 \end{array}\right]$

$v_2 = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

$x = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

$$x = c_1 e^{(-1+i)t}\begin{bmatrix} 2+i \\ 1 \end{bmatrix} + c_2 e^{(-1-i)t}\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

$$z_1(t) = e^{(-1+i)t}\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

$$= e^{-t}(-\cos t + i\sin t)\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$= e^{-t}\left(\cos t\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + ie^{-t}\left(\cos t\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

$$= e^{-t}\begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + ie^{-t}\begin{bmatrix} -\cos t - 2\sin t \\ \sin t \end{bmatrix}$$

real

imaginary

$$\boxed{\vec{x} = c_1 e^{-t}\begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -\cos t + 2\sin t \\ \sin t \end{bmatrix}}$$

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Problem 4

a) $y' = (y+1)(y^2-a)$ $\quad a \in \mathbb{R}, a > 1$

$y' = (y+1)(y-\sqrt{a})(y+\sqrt{a})$

$y' = 0$ when $y = -1, \sqrt{a}$, or $-\sqrt{a}$

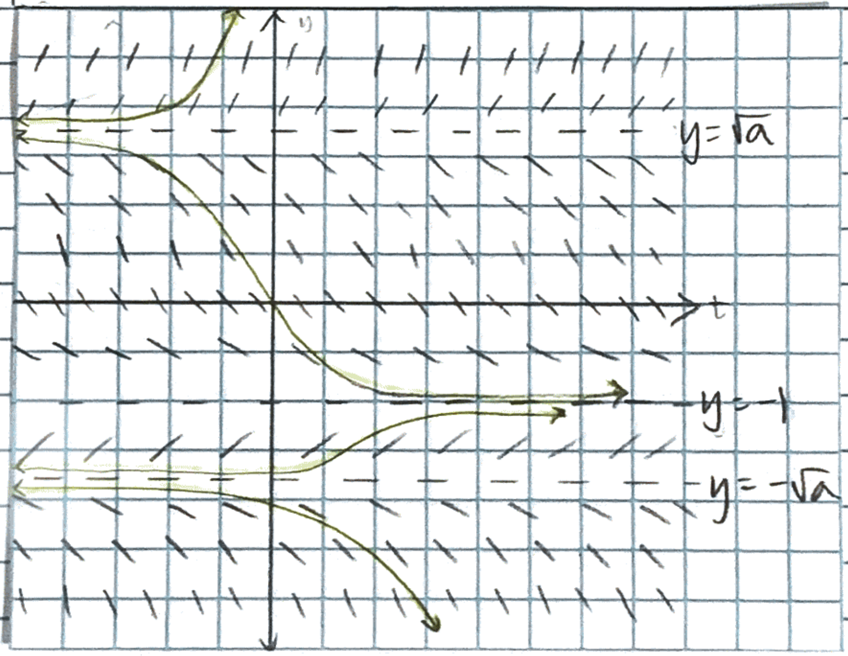
if $a > 1$, then $\sqrt{a} > 1$ and $-\sqrt{a} < -1$

$-\sqrt{a} \quad -1 \quad \sqrt{a}$

if $y > \sqrt{a}$, $y' = +$
if $y < \sqrt{a}$ and $y > -1$, $y' = -$
if $y < -1$ and $y > -\sqrt{a}$, $y' = +$
if $y < -\sqrt{a}$, $y' = -$

$\boxed{\begin{array}{l} y = -1 \text{ is asymptotically stable} \\ y = -\sqrt{a} \text{ and } y = \sqrt{a} \text{ are unstable} \end{array}}$

b)



c) $y' = (y+1)(y^2 - a) \qquad y(0) = y_0$

if $y_0 < -\sqrt{a}$:

$\lim_{t \to \infty} y(t) = -\infty$

if $y_0 > -\sqrt{a}$ & $y_0 < -1$

$\lim_{t \to \infty} y(t) = -1$

if $y_0 > -1$ & $y_0 < \sqrt{a}$

$\lim_{t \to \infty} y(t) = -1$

if $y_0 > \sqrt{a}$

$\lim_{t \to \infty} y(t) = \infty$

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problem 5

$(2t^3 - 6t^2y + 3ty^2)dt + (-2t^3 + kt^2y - y^3)dy = 0$

a) For this equation to be exact

$\frac{\partial}{\partial y}(2t^3 - 6t^2y + 3ty^2) = \frac{\partial}{\partial t}(-2t^3 + kt^2y - y^3)$

$-6t^2 + 6ty = -6t^2 + 2kyt$

$6ty = 2kyt$

$\boxed{k=3}$

b) $(2t^3 - 6t^2y + 3ty^2)dt + (-2t^3 + 3t^2y - y^3)dy = 0$

$\frac{\partial}{\partial t}F(t,y) = 2t^3 - 6t^2y + 3ty^2$

so $F(t,y) = \frac{1}{2}t^4 - 2t^3y + \frac{3t^2y^2}{2} + \varphi(y) = C_1$

$\frac{\partial}{\partial y}F(t,y) = -2t^3 + 3t^2y + \varphi'(y)$

so $\varphi'(y) = -y^3$

$\varphi(y) = \frac{-y^4}{4} + C_2$ let $C = C_1 - C_2$

$$\boxed{F(t,y) = \frac{1}{2}t^4 - 2t^3y + \frac{3t^2y^2}{2} - \frac{y^4}{4} = C}$$

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Problem 6

$A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}$

a) find all eigenvalues and basis of each eigenspace

$p(\lambda) = \det(A - \lambda I)$

$= \det \begin{bmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{bmatrix}$

$= (1-\lambda)(-7-\lambda) + 16$

$= -7 - 6\lambda + \lambda^2 + 16$

$= \lambda^2 - 6\lambda + 9$

$p(\lambda) = (\lambda - 3)^2 = 0$

$\boxed{\lambda = 3 \text{, with multiplicity } 2}$

E_3:

$(A - 3I)v = 0$

$\begin{bmatrix} -2 & -4 \\ 4 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} -2 & -4 & 0 \\ 4 & -10 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -18 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{E_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}}$

b) $\boxed{\text{no}}$, no eigenbasis

c) $\vec{x}' = A\vec{x}$

$\vec{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{bmatrix} \rightarrow \lambda = -3$ with multiplicity 2

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ need v_2, a generalized eigenvector

$(A-\lambda I)v_2 = v_1 \qquad \lambda = -3$

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & -4 & 1 \\ 4 & -4 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{4} \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = \frac{1}{4} + x_2$

$x_2 =$ free $\rightarrow$ let $x_2 = 0$

$$v_2 = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}$$

general solution: $\vec{x} = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t}(v_2 + t v_1)$

$$\boxed{\vec{x} = c_1 e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\left(\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)}$$

d) if $\vec{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + \frac{1}{4}c_2 \\ c_1 + 0 \end{bmatrix}$$

$c_1 = 2, \quad c_2 = 4$

$$\boxed{\vec{x} = 2e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4e^{-3t}\left(\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)}$$

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Problem 7

1) $2y'' + y' - y = 3e^{-t} + 2e^{-\frac{t}{2}}$

$p(\lambda) = 2\lambda^2 + \lambda - 1 = 0$

$(\lambda+1)(2\lambda-1) = 0$

$\lambda = -1, \frac{1}{2}$

homogenous equation's fundamental set: $y_1 = e^{-t}$, $y_2 = e^{\frac{t}{2}}$

$y_h(t) = C_1 e^{-t} + C_2 e^{\frac{t}{2}}$

variation of parameters

$2y'' + y' - y = 3e^{-t} + 2e^{-\frac{t}{2}}$

$y'' + \frac{y'}{2} - \frac{y}{2} = \frac{3e^{-t} + 2e^{-\frac{t}{2}}}{2}$

$v_1' y_1 + v_2' y_2 = 0$ $\quad$ $v_1' y_1' + v_2' y_2' = \frac{3e^{-t} + 2e^{-\frac{t}{2}}}{2}$

$v_1'(e^{-t}) + v_2'(e^{\frac{t}{2}}) = 0$

$v_1'(-e^{-t}) + v_2'\left(\frac{1}{2}e^{\frac{t}{2}}\right) = \frac{3e^{-t} + 2e^{-\frac{t}{2}}}{2}$

$v_2' \frac{3}{2} e^{\frac{t}{2}} = \frac{3e^{-t} + 2e^{-\frac{t}{2}}}{2}$

$3v_2' e^{\frac{t}{2}} = 3e^{-t} + 2e^{-\frac{t}{2}}$

$3v_2' = 3e^{\frac{3t}{2}} + 2e^{-t}$

$v_2' = e^{-\frac{3t}{2}} + \frac{2}{3}e^{-t}$

$v_2 = -\frac{2}{3}e^{-\frac{3t}{2}} - \frac{2}{3}e^{-t}$

$v_1'(e^{-t}) + (e^{\frac{t}{2}})\left(e^{-\frac{3t}{2}} + \frac{2}{3}e^{-t}\right) = 0$

$v_1'(e^{-t}) = -\left(e^{-t} + \frac{2}{3}e^{-\frac{t}{2}}\right)$

$v_1' = -\left(1 + \frac{2}{3}e^{\frac{t}{2}}\right) = -1 - \frac{2}{3}e^{\frac{t}{2}}$

$$v_1 = -t - \frac{4}{3}e^{\frac{t}{2}}$$

$$y_p = v_1 y_1 + v_2 y_2$$
$$= \left(-t - \frac{4}{3}e^{\frac{t}{2}}\right)(e^{-t}) + \left(-\frac{2}{3}e^{-\frac{3t}{2}} - \frac{2}{3}e^{-t}\right)(e^{\frac{t}{2}})$$
$$= -te^{-t} - \frac{4}{3}e^{-\frac{t}{2}} - \frac{2}{3}e^{-t} - \frac{2}{3}e^{-\frac{t}{2}}$$
$$= -te^{-t} - 2e^{-\frac{t}{2}} - \frac{2}{3}e^{-t}$$

$$y_h = C_1 e^{-t} + C_2 e^{\frac{t}{2}}$$

$$y(t) = \underline{C_1 e^{-t}} + C_2 e^{\frac{t}{2}} - te^{-t} - 2e^{-\frac{t}{2}} \underline{- \frac{2}{3}e^{-t}}$$

$$\boxed{y(t) = C_1 e^{-t} + C_2 e^{\frac{t}{2}} - te^{-t} - 2e^{-\frac{t}{2}}}$$

2) $y(0) = 2$ and $y(t)$ does not approach $\pm\infty$ when $t \to \infty$

$$y(0) = C_1 + C_2 - 0 - 2 = 2$$
$$C_1 + C_2 = 4$$

for $y(t)$ to not approach $\pm\infty$ when $t \to \infty$, $y'(t) = 0$ as $t \to \infty$

$$y'(t) = -C_1 e^{-t} + \frac{C_2}{2}e^{\frac{t}{2}} - (-te^{-t} + e^{-t}) + e^{-\frac{t}{2}} = 0$$
$$= -C_1 e^{-t} + \frac{C_2}{2}e^{\frac{t}{2}} + te^{-t} - e^{-t} + e^{-\frac{t}{2}} = 0$$

no unique solution — the second condition cannot be satisfied catherine tta

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Problem 8

$t^2y'' - 3ty' + 3y = 12t^4 \qquad t>0$

1) $y(t) = t^\lambda$

$y'(t) = \lambda t^{\lambda-1}$

$y''(t) = \lambda(\lambda-1)t^{\lambda-2}$

$t^2y'' - 3ty' + 3y = 0$ homogeneous equation

$t^2(\lambda)(\lambda-1)t^{\lambda-2} - 3t(\lambda)(t^{\lambda-1}) + 3t^\lambda = 0$

$(\lambda^2-\lambda)t^\lambda - 3\lambda t^\lambda + 3t^\lambda = 0$

$t^\lambda(\lambda^2 - 4\lambda + 3) = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda-3)(\lambda-1) = 0$

$\lambda_1 = 1, \ \lambda_2 = 3$

$y(t) = C_1y_1(t) + C_2y_2(t)$

given $y(t) = t^\lambda$, $y_1(t) = t$ and $y_2(t) = t^3$

$\boxed{y_h(t) = C_1 t + C_2 t^3}$

2) $t^2y'' - 3ty' + 3y = 12t^4 \qquad y_1 = t, \ y_2 = t^3$

$y'' - \frac{3y'}{t} + \frac{3y}{t^2} = 12t^2$

$v_1'y_1 + v_2'y_2 = 0$

$v_1'(t) + v_2'(t^3) = 0$

$tv_1' + t^3v_2' = 0$

$v_1' + t^2v_2' = 0$

$v_1' = -6t^2$

$v_1 = -2t^3$

$v_1'y_1' + v_2'y_2' = 12t^2$

$v_1'(1) + v_2'(3t^2) = 12t^2$

$v_1' + 3t^2v_2' = 12t^2$

$2t^2v_2' = 12t^2$

$v_2' = 6$

$v_2 = 6t$

$y_p = v_1y_1 + v_2y_2$

$y_p = (-2t^3)(t) + (t^3)(6t)$

$y_p = -2t^4 + 6t^4$

$$\boxed{y_p(t) = 4t^4}$$

c) general solution

↳ $y_h + y_p$

$$\boxed{y(t) = C_1 t + C_2 t^3 + 4t^4}$$