# 21W-MATH33B-1 Final Exam

ERIC YANG

TOTAL POINTS

## 100 / 100

QUESTION 1

**1 10 / 10**

✓ **- 0 pts** Correct

**- 1 pts** Click here to replace this description.

**- 5 pts** Click here to replace this description.

**- 10 pts** Click here to replace this description.

**- 2 pts** Click here to replace this description.

**- 3 pts** Click here to replace this description.

QUESTION 2

**2 12 / 12**

✓ **- 0 pts** Correct

**- 1 pts** Click here to replace this description.

**- 2 pts** Click here to replace this description.

**- 12 pts** Click here to replace this description.

QUESTION 3

**3 12 / 12**

✓ **- 0 pts** Correct

**- 2 pts** Find characteristic polynomial

**- 2 pts** Find eigenvalues

**- 2 pts** Find eigenvectors

**- 2 pts** Find the general solution

**- 1 pts** Minor error

QUESTION 4

**4 15 / 15**

✓ **- 0 pts** Correct

**- 2 pts** Identify equilibria

**- 2 pts** Correct phase line

**- 2 pts** Correct stability

**- 2 pts** Good plot

**- 2 pts** Correct limits

**- 1 pts** Minor issue

QUESTION 5

**5 12 / 12**

✓ **+ 12 pts** Correct

**+ 10 pts** Minor error

**+ 10 pts** Equation not in correct form

**+ 0 pts** Incorrect

QUESTION 6

**6 12 / 12**

✓ **+ 12 pts** Correct

**+ 3 pts** Part a

**+ 3 pts** Part b

**+ 3 pts** Part c

**+ 3 pts** Part d

**+ 0 pts** Incorrect

QUESTION 7

**7 15 / 15**

✓ **+ 15 pts** Correct

**+ 3 pts** Found characteristic polynomial $$p(\lambda)=\lambda^2 +\frac{\lambda}{2}-\frac{1}{2}=(\lambda -\frac{1}{2})(\lambda +1)$$ for the homogeneous equation, and used it to find the general solution $$y(1)=C_1 e^{\frac{1}{2} t} + C_2 e^{-t}$$ to the homogeneous equation

**+ 8 pts** Used undetermined coefficients to solve $$2 y'' +y' -y =3 e^{-t}$$ and $$2 y'' +y' -y =2 e^{-\frac{1}{2} t}$$, getting particular solutions $$y(t)=-t e^{-t}$$ and $$y(t)=-2 e^{-\frac{1}{2} t}$$, to get the general solution $$y(t) = C_1 e^{\frac{1}{2} t} + C_2 e^{-t} -t e^{-t}-2 e^{-\frac{1}{2} t} $$

**+ 8 pts** Alternatively, used variation of parameters with $$y_1 =e^{\frac{1}{2}t}$$, $$y_2 =e^{-t}$$, $$g=\frac{3}{2} e^{-t}+e^{-\frac{1}{2}t}$$ to get a particular solution $$y_p =-\frac{2}{3} e^{-t}-2e^{-\frac{1}{2} t}-te^{-t}$$ and thence the general solution $$y(t) = C_1 e^{\frac{1}{2} t} + C_2 e^{-t} -\frac{2}{3} e^{-

t}-2e^{-\frac{1}{2} t}-te^{-t}$$

+ **4 pts** Found that the desired solution is unique $$y=4e^{-t} -2e^{-\frac{1}{2} t} -te^{-t}$$

QUESTION 8

**8 12 / 12**

✓ **+ 12 pts** **Correct**

+ **4 pts** Substituted $$y(t)=t^\lambda$$ in the homogeneous equation to get $$\lambda^2 -4\lambda +3=0$$, so $$\lambda \in \{ 1,3 \}$$.

+ **4.8 pts** Remember that you need to divide by $$t^2$$, so that the coefficient of $$y''$$ is 1 and $$g(t)=12t^2$$.  A common mistake was to forget to do this, giving something like $$y_p =\frac{4}{5}t^6$$ instead of the correct answer $$y_p =4t^4$$

+ **2 pts** Added the particular solution to the general solution to the homogeneous equation (to get $$y=4 t^4 +C_1 t + C_2 t^3$$, if everything else was right)

+ **6 pts** Took $$y_1 (t)=t , y_2 (t)=t^3$$; the Wronskian is $$W=t\cdot (3t^2) -t^3 \cdot 1 = 2t^3$$, so $$v_1 = \int \frac{-12t^5}{2t^3} dt =-2t^3$$, $$v_2 =\int \frac{12 t^3}{2 t^3} dt =6t$$, giving a particular solution $$y = v_1 y_1 + v_2 y_2 =4t^4$$

Eric Yang
ID# 245565249

Final Math 33B

1) $y' + ay = t^n e^{-at}$ $\quad a \in \mathbb{R} \quad n \in \mathbb{N}$

$\mu(t) = e^{\int a\,dt}$

$= e^{at}$

$e^{at} y' + a e^{at} y = t^n$

$(e^{at} y)' = t^n$

$e^{at} y = \int t^n \, dt$

$= \frac{t^{n+1}}{n+1} + C$

$$\boxed{y = \frac{t^{n+1}}{e^{at}(n+1)} + \frac{C}{e^{at}}, \; C \in \mathbb{R}}$$

2a) $y''' - 4y'' - 7y' + 10y = 0$

let $y_1'(t) = y_2(t)$, $y_2'(t) = y_3(t)$

$y_3'(t) = 4y'' + 7y' - 10y = 4y_3(t) + 7y_2(t) - 10y_1(t)$

$$\boxed{\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & 7 & 4 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}}$$

2b) $p(\lambda) = \det(A - \lambda I) = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -10 & 7 & 4-\lambda \end{bmatrix}$

$= -\lambda \begin{bmatrix} -\lambda & 1 \\ 7 & 4-\lambda \end{bmatrix} - 1 \begin{bmatrix} 0 & 1 \\ -10 & 4-\lambda \end{bmatrix}$

$= -\lambda(-\lambda(4-\lambda) - 7) - (0 + 10)$

$= -\lambda^3 + 4\lambda^2 + 7\lambda - 10$

$= (\lambda - 1)(\lambda + 2)(\lambda - 5)$

$\lambda_1 = 1, \; \lambda_2 = -2, \; \lambda_3 = 5$

$\lambda_1 = 1: \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -10 & 7 & 3 \end{bmatrix} \begin{matrix} -① - ② \\ \\ \\ \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -10 & 7 & 3 \end{bmatrix} \begin{matrix} -② \\ ③ + 10① - 7② \\ \\ \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $y_3$ free

$y_1 - y_3 = 0 \quad y_2 - y_3 = 0 \quad$ let $y_3 = 1 \quad y_1 = 1, y_2 = 1$

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

1 **10 / 10**

✓ **- 0 pts** Correct

**- 1 pts** Click here to replace this description.

**- 5 pts** Click here to replace this description.

**- 10 pts** Click here to replace this description.

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$e^{at} y' + a e^{at} y = t^n$
$(e^{at} y)' = t^n$
$e^{at} y = \int t^n dt$
$= \frac{t^{n+1}}{n+1} + C$

$$\boxed{y = \frac{t^{n+1}}{e^{at}(n+1)} + \frac{C}{e^{at}}, \ C \in \mathbb{R}}$$

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2b) $p(\lambda) = \det(A - \lambda I) = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -10 & 7 & 4-\lambda \end{bmatrix}$

$= -\lambda \begin{bmatrix} -\lambda & 1 \\ 7 & 4-\lambda \end{bmatrix} - 1 \begin{bmatrix} 0 & 1 \\ -10 & 4-\lambda \end{bmatrix}$

$= -\lambda(-\lambda(4-\lambda) - 7) - (0 + 10)$

$= -\lambda^3 + 4\lambda^2 + 7\lambda - 10$

$= (\lambda - 1)(\lambda + 2)(\lambda - 5)$

$\lambda_1 = 1, \ \lambda_2 = -2, \ \lambda_3 = 5$

$\lambda_1 = 1 : \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -10 & 7 & 3 \end{bmatrix} \begin{matrix} -①-② \\ \\ \\ \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -10 & 7 & 3 \end{bmatrix} \begin{matrix} -② \\ ③+10①-7② \\ \\ \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $y_3$ free

$y_1 - y_3 = 0 \quad y_2 - y_3 = 0 \quad$ let $y_3 = 1 \quad y_1 = 1, y_2 = 1$

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -2$ $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -10 & 7 & 6 \end{bmatrix}$ ① × 0.5 − 0.25 ② $\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 2 & 1 \\ -10 & 7 & 6 \end{bmatrix}$ ② × 0.5 $\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ -10 & 7 & 6 \end{bmatrix}$

3 + 10 ① − 7 ② $\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ $y_1 = +\frac{1}{4}y_3$ $y_2 = -\frac{1}{2}y_3$ $y_3$ free

$y_3 = 4$ $y_1 = 1$ $y_2 = -2$ $v_2 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

$\lambda_3 = 5$ $\begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ -10 & 7 & -1 \end{bmatrix}$ ① × −0.2 ② × −0.2 $\Rightarrow \begin{bmatrix} 1 & -0.2 & 0 \\ 0 & 1 & -0.2 \\ -10 & 7 & -1 \end{bmatrix}$ ① + 0.2 ② ③ + 10 ① − 7 ② $\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{25} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}$

$y_3$ free

$y_1 = \frac{1}{25}y_3$ $y_2 = \frac{1}{5}y_3$ $y_3 = 25$ $y_1 = 1$ $y_2 = 5$

$v_3 = \begin{pmatrix} 1 \\ 5 \\ 25 \end{pmatrix}$

$$v = C_1 e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + C_3 e^{5t} \begin{bmatrix} 1 \\ 5 \\ 25 \end{bmatrix}$$

$$\boxed{v = \begin{bmatrix} C_1e^{t} + C_2e^{-2t} + C_3e^{5t} \\ C_1e^{t} - 2C_2e^{-2t} + 5C_3e^{5t} \\ C_1e^{t} + 4C_2e^{-2t} + 25C_3e^{5t} \end{bmatrix}}$$

c) $v_1(t) = \boxed{y(t) = C_1e^{t} + C_2e^{-2t} + C_3e^{5t}}$

3) $p(\lambda) = \det \begin{bmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{bmatrix}$

$= (1-\lambda)(-3-\lambda) + 5$

$= \lambda^2 + 2\lambda - 3 + 5 = \lambda^2 + 2\lambda + 2$

$= (\lambda+1)^2 + 1 = 0$ $\therefore \lambda = -1 \pm i$

$\lambda_1 = -1 - i$ $\begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2+i \\ 2+i & -5 \end{bmatrix}$ ② − (2+i) ① $\Rightarrow \begin{bmatrix} 1 & -2+i \\ 0 & 0 \end{bmatrix}$ $x_2$ free

$v_1 = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

$\lambda_2 = -1 + i$ $\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2-i \\ 2-i & -5 \end{bmatrix}$ ② − (2−i) ① $\Rightarrow \begin{bmatrix} 1 & -2-i \\ 0 & 0 \end{bmatrix}$ $x_2$ free

$v_2 = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$

**2 12 / 12**

- ✓ **- 0 pts** Correct
- **- 1 pts** Click here to replace this description.
- **- 2 pts** Click here to replace this description.
- **- 12 pts** Click here to replace this description.

$\lambda_2 = -2$ $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -10 & 7 & 6 \end{bmatrix}$ ① × 0.5 − 0.25 ② $\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 2 & 1 \\ -10 & 7 & 6 \end{bmatrix}$ ② × 0.5 $\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ -10 & 7 & 6 \end{bmatrix}$

3 + 10 ① − 7 ② $\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ $y_1 = +\frac{1}{4} y_3$ $y_2 = -\frac{1}{2} y_3$ $y_3$ free

$y_3 = 4$ $y_1 = 1$ $y_2 = -2$ $v_2 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

$\lambda_3 = 5$ $\begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ -10 & 7 & -1 \end{bmatrix}$ ① × −0.2, ② × −0.2 $\Rightarrow \begin{bmatrix} 1 & -0.2 & 0 \\ 0 & 1 & -0.2 \\ -10 & 7 & -1 \end{bmatrix}$ ① + 0.2 ②, ③ + 10 ① − 7 ② $\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{25} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}$

$y_3$ free

$y_1 = \frac{1}{25} y_3$ $y_2 = \frac{1}{5} y_3$ $y_3 = 25$ $y_1 = 1$ $y_2 = 5$

$v_3 = \begin{pmatrix} 1 \\ 5 \\ 25 \end{pmatrix}$

$$v = C_1 e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + C_3 e^{5t} \begin{bmatrix} 1 \\ 5 \\ 25 \end{bmatrix}$$

$$\boxed{v = \begin{bmatrix} C_1 e^{t} + C_2 e^{-2t} + C_3 e^{5t} \\ C_1 e^{t} - 2C_2 e^{-2t} + 5C_3 e^{5t} \\ C_1 e^{t} + 4C_2 e^{-2t} + 25 C_3 e^{5t} \end{bmatrix}}$$

c) $\cancel{v_1(t)}$ $\boxed{y(t) = C_1 e^{t} + C_2 e^{-2t} + C_3 e^{5t}}$

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$= (1-\lambda)(-3-\lambda) + 5$

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$\lambda_1 = -1 - i$ $\begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix} \circlearrowright \begin{bmatrix} 1 & -2+i \\ 2+i & -5 \end{bmatrix}$ ② − (2+i) ① $\Rightarrow \begin{bmatrix} 1 & -2+i \\ 0 & 0 \end{bmatrix}$ $x_2$ free

$v_1 = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

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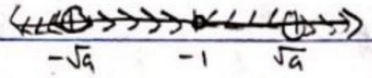
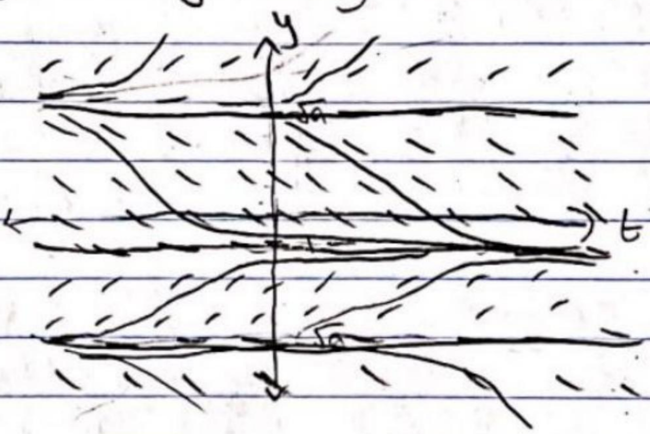
$v_2 = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$

$$z_1(t)=e^{(-1+i)t}\begin{bmatrix}2+i\\1\end{bmatrix}$$
$$=e^{-t}(\cos(t)+i\sin(t))\left(\begin{bmatrix}2\\1\end{bmatrix}+i\begin{bmatrix}1\\0\end{bmatrix}\right)$$
$$=e^{-t}\left(\cos(t)\begin{bmatrix}2\\1\end{bmatrix}-\sin(t)\begin{bmatrix}1\\0\end{bmatrix}\right)+ie^{-t}\left(\cos(t)\begin{bmatrix}1\\0\end{bmatrix}+\sin(t)\begin{bmatrix}2\\1\end{bmatrix}\right)$$
$$=e^{-t}\begin{bmatrix}2\cos(t)-\sin(t)\\ \cos(t)\end{bmatrix}+ie^{-t}\begin{bmatrix}\cos(t)+2\sin(t)\\ \sin(t)\end{bmatrix}$$
$$x(t)=\boxed{C_1e^{-t}\begin{bmatrix}2\cos(t)-\sin(t)\\ \cos(t)\end{bmatrix}+C_2e^{-t}\begin{bmatrix}\cos(t)+2\sin(t)\\ \sin(t)\end{bmatrix}}$$

4a) 


$y=\pm\sqrt{a}$ unstable eq. point
$y=-1$ asymptotically stable eq. point

4b) 

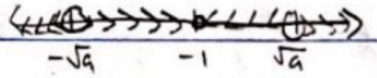

4c)
$$\lim_{t\to\infty}y(t)=\begin{cases}\infty & y_0>\sqrt{a}\\ -1 & -\sqrt{a}<y_0<\sqrt{a}\\ -\infty & y_0<-\sqrt{a}\\ \sqrt{a} & y_0=\sqrt{a}\\ -\sqrt{a} & y_0=-\sqrt{a}\end{cases}$$

5a) $(2t^3-6t^2y+3ty^2)\,dt+(-2t^3+kt^2y-y^3)\,dy=0$
$\frac{\partial}{\partial y}(2t^3-6t^2y+3ty^2)=-6t^2+6ty$
$\frac{\partial}{\partial t}(-2t^3+kt^2y-y^3)=-6t^2+2kty \qquad 2k=6,\ \boxed{k=3}$

3 **12 / 12**

- ✓ **- 0 pts** Correct
- **- 2 pts** Find characteristic polynomial
- **- 2 pts** Find eigenvalues
- **- 2 pts** Find eigenvectors
- **- 2 pts** Find the general solution
- **- 1 pts** Minor error

$$z_1(t) = e^{(-1+i)t}\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$
$$= e^{-t}(\cos(t) + i\sin(t))\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$
$$= e^{-t}\left(\cos(t)\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \sin(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + ie^{-t}\left(\cos(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(t)\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$
$$= e^{-t}\begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + ie^{-t}\begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{bmatrix}$$
$$x(t) = \boxed{C_1 e^{-t}\begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{bmatrix}}$$

4a) 

$y = \pm\sqrt{a}$ unstable eq. point
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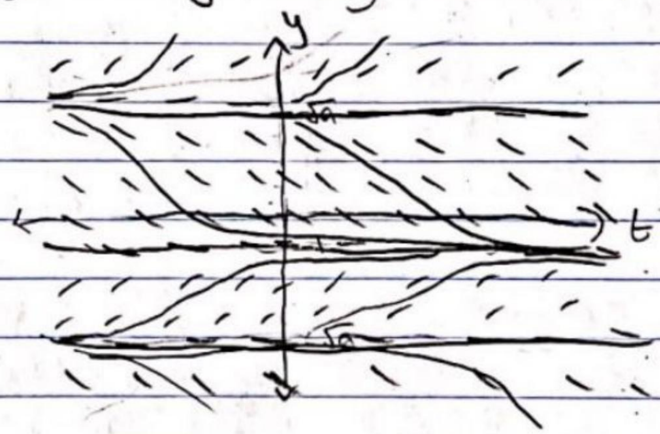
4b) 

4c)
$$\lim_{t\to\infty} y(t) = \begin{cases} \infty & y_0 > \sqrt{a} \\ -1 & -\sqrt{a} < y_0 < \sqrt{a} \\ -\infty & y_0 < -\sqrt{a} \\ \sqrt{a} & y_0 = \sqrt{a} \\ -\sqrt{a} & y_0 = -\sqrt{a} \end{cases}$$

5a) $(2t^3 - 6t^2y + 3ty^2)\,dt + (-2t^3 + kt^2y - y^3)\,dy = 0$
$\frac{\partial}{\partial y}(2t^3 - 6t^2y + 3ty^2) = -6t^2 + 6ty$
$\frac{\partial}{\partial t}(-2t^3 + kt^2y - y^3) = -6t^2 + 2kty$ $\quad 2k = 6$, $\boxed{k = 3}$

4 **15 / 15**

✓ **- 0 pts** Correct

**- 2 pts** Identify equilibria

**- 2 pts** Correct phase line

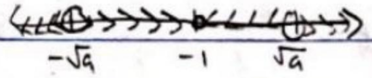
**- 2 pts** Correct stability

**- 2 pts** Good plot

**- 2 pts** Correct limits

**- 1 pts** Minor issue

$$z_1(t) = e^{(-1+i)t}\begin{bmatrix}2+i\\1\end{bmatrix}$$
$$= e^{-t}(\cos(t) + i\sin(t))\left(\begin{bmatrix}2\\1\end{bmatrix} + i\begin{bmatrix}1\\0\end{bmatrix}\right)$$
$$= e^{-t}\left(\cos(t)\begin{bmatrix}2\\1\end{bmatrix} - \sin(t)\begin{bmatrix}1\\0\end{bmatrix}\right) + ie^{-t}\left(\cos(t)\begin{bmatrix}1\\0\end{bmatrix} + \sin(t)\begin{bmatrix}2\\1\end{bmatrix}\right)$$
$$= e^{-t}\begin{bmatrix}2\cos(t) - \sin(t)\\ \cos(t)\end{bmatrix} + ie^{-t}\begin{bmatrix}\cos(t) + 2\sin(t)\\ \sin(t)\end{bmatrix}$$
$$x(t) = \boxed{C_1 e^{-t}\begin{bmatrix}2\cos(t) - \sin(t)\\ \cos(t)\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}\cos(t) + 2\sin(t)\\ \sin(t)\end{bmatrix}}$$

4a) 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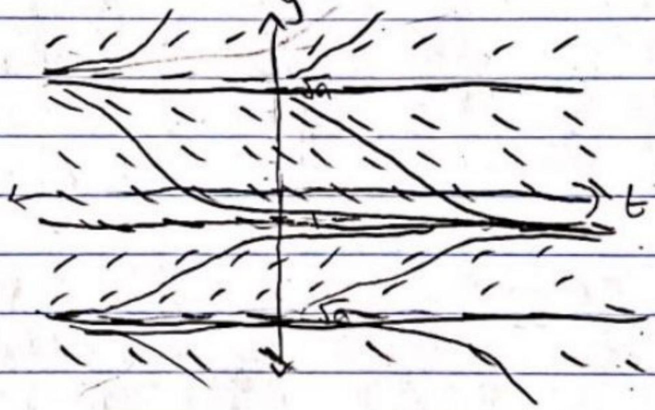
4b) 


4c)
$$\lim_{t\to\infty} y(t) = \begin{cases} \infty & y_0 > \sqrt{a} \\ -1 & -\sqrt{a} < y_0 < \sqrt{a} \\ -\infty & y_0 < -\sqrt{a} \\ \sqrt{a} & y_0 = \sqrt{a} \\ -\sqrt{a} & y_0 = -\sqrt{a} \end{cases}$$

5a) $(2t^3 - 6t^2y + 3ty^2)\,dt + (-2t^3 + kt^2y - y^3)\,dy = 0$
$$\frac{\partial}{\partial y}(2t^3 - 6t^2y + 3ty^2) = -6t^2 + 6ty$$
$$\frac{\partial}{\partial t}(-2t^3 + kt^2y - y^3) = -6t^2 + 2kty \qquad 2k = 6,\ \boxed{k = 3}$$

8b) $F(t,y) = \int (2t^3 - 6t^2 y + 3ty^2)\,dt + \phi(y)$

$F(t,y) = \frac{t^4}{2} - 2t^3 y + \frac{3}{2}t^2 y^2 + \phi(y)$

$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\frac{t^4}{2} - 2t^3 y + \frac{3}{2}t^2 y^2 + \phi(y)\right)$

$= -2t^3 + 3t^2 y + \phi'(y)$ $\qquad \phi'(y) = -y^3$

$\phi_y = -\frac{y^4}{4} + C$

$F(t,y) = \frac{t^4}{2} - 2t^3 y + \frac{3}{2}t^2 y^2 - \frac{y^4}{4} + C$

$\boxed{\frac{t^4}{2} - 2t^3 y + \frac{3}{2}t^2 y^2 - \frac{y^4}{4} = C}$

6a) $\det(A - \lambda I) = 0$

$(1-\lambda)(-7-\lambda) + 16 = 0$

$\lambda^2 + 6\lambda - 7 + 16 = 0$

$\lambda^2 + 6\lambda + 9 = 0$

$(\lambda+3)^2 = 0 \qquad \lambda = \boxed{-3}$ multiplicity 2

$\lambda = -3 \quad \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{array}{l} ① \div 4 \\ ② - ① \end{array} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x_2$ free

$x_1 = x_2 \quad x_2 = x_2 \quad E_1 = \text{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A - \lambda_1 I)v_2 = v_1$

b) $\boxed{\text{No}}$ $\quad \left[\begin{array}{cc|c} 4 & -4 & 1 \\ 4 & -4 & 1 \end{array}\right] \begin{array}{l} ① \div 4 \\ ② - ① \end{array} \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{4} \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = \frac{1}{4} \\ x_2 = 0 \end{array} \quad v_2 = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$

c) $\quad x_2 = e^{-3t}\left(\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{e^{-3t}}{4} + te^{-3t} \\ te^{-3t} \end{bmatrix}$

$x(t; C_1 C_2) = C_1 e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3t}\left(\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$= \boxed{\begin{bmatrix} C_1 e^{-3t} + C_2 e^{-3t}(\frac{1}{4} + t) \\ C_1 e^{-3t} + C_2 e^{-3t} t \end{bmatrix}}$

6d) $x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} C_1 + \frac{1}{4}C_2 \\ C_1 \end{bmatrix} \qquad C_1 = 2 \quad C_2 = 4$

$x(t) = \boxed{\begin{bmatrix} 2e^{-3t} + 4e^{-3t}(\frac{1}{4} + t) \\ 2e^{-3t} + 4e^{-3t} t \end{bmatrix}}$

5 **12 / 12**

✓ **+ 12 pts** Correct

**+ 10 pts** Minor error

**+ 10 pts** Equation not in correct form

**+ 0 pts** Incorrect

8b) $F(t,y) = \int (2t^3 - 6t^2y + 3ty^2)\,dt + \phi(y)$

$F(t,y) = \frac{t^4}{2} - 2t^3y + \frac{3}{2}t^2y^2 + \phi(y)$

$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\frac{t^4}{2} - 2t^3y + \frac{3}{2}t^2y^2 + \phi(y)\right)$

$= -2t^3 + 3t^2y + \phi'(y) \qquad \phi'(y) = -y^3$

$\phi_y = -\frac{y^4}{4} + C$

$F(t,y) = \frac{t^4}{2} - 2t^3y + \frac{3}{2}t^2y^2 - \frac{y^4}{4} + C$

$\boxed{\frac{t^4}{2} - 2t^3y + \frac{3}{2}t^2y^2 - \frac{y^4}{4} = C}$

6a) $\det(A - \lambda I) = 0$

$(1-\lambda)(-7-\lambda) + 16 = 0$

$\lambda^2 + 6\lambda - 7 + 16 = 0$

$\lambda^2 + 6\lambda + 9 = 0$

$(\lambda+3)^2 = 0 \qquad \lambda = \boxed{-3}$ multiplicity 2

$\lambda = -3 \quad \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{array}{l} ① \div 4 \\ ② - ① \end{array} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x_2$ free

$x_1 = x_2 \quad x_2 = x_2 \quad E_1 = \text{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A - \lambda_1 I)v_2 = v_1$

b) $\boxed{No}$ $\quad \left[\begin{array}{cc|c} 4 & -4 & 1 \\ 4 & -4 & 1 \end{array}\right] \begin{array}{l} ① \div 4 \\ ② - ① \end{array} \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{4} \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = \frac{1}{4} \\ x_2 = 0 \end{array} \quad v_2 = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$

c) $\quad x_2 = e^{-3t}\left(\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{e^{-3t}}{4} + te^{-3t} \\ te^{-3t} \end{bmatrix}$

$x(t; C_1 C_2) = C_1e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2e^{-3t}\left(\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$= \boxed{\begin{bmatrix} C_1e^{-3t} + C_2e^{-3t}(\frac{1}{4} + t) \\ C_1e^{-3t} + C_2e^{-3t}t \end{bmatrix}}$

6d) $x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} C_1 + \frac{1}{4}C_2 \\ C_1 \end{bmatrix} \qquad C_1 = 2 \quad C_2 = 4$

$x(t) = \boxed{\begin{bmatrix} 2e^{-3t} + 4e^{-3t}(\frac{1}{4} + t) \\ 2e^{-3t} + 4e^{-3t}t \end{bmatrix}}$

**6 12 / 12**

- ✓ **+ 12 pts** Correct
- **+ 3 pts** Part a
- **+ 3 pts** Part b
- **+ 3 pts** Part c
- **+ 3 pts** Part d
- **+ 0 pts** Incorrect

7a) $2y'' + y' - y = 3e^{-t} + 2e^{-t/2}$

$p(\lambda) = 2\lambda^2 + \lambda - 1$

$= (2\lambda - 1)(\lambda + 1) \quad \lambda = \frac{1}{2}, -1$

$g(t) = 3e^{-t} + 2e^{-t/2}$

$y_p(t) = Ate^{-t} + Be^{-t/2}$, since $e^{-t}$ is already a solution

$y_p'(t) = -Ate^{-t} + Ae^{-t} - \frac{1}{2}Be^{-t/2}$

$y_p''(t) = Ate^{-t} - 2Ae^{-t} + \frac{1}{4}Be^{-t/2}$

$2y_p''(t) + y_p'(t) - y_p(t) = 2Ate^{-t} - 4Ae^{-t} + \frac{1}{2}Be^{-t/2} - Ate^{-t} - \frac{1}{2}Be^{-t/2} - Ate^{-t} - Be^{-t/2}$

$= -3Ae^{-t} - Be^{-t/2}$

$-3Ae^{-t} - Be^{-t/2} = 3e^{-t} + 2e^{-t/2}$

$A = -1, \; B = -2$

$y_p(t) = -te^{-t} - 2e^{-t/2}$

$y = \boxed{C_1 e^{t/2} + C_2 e^{-t} - te^{-t} - 2e^{-t/2}}$

7b) $y(0) = 2 = C_1 + C_2 - 2$

$C_1 + C_2 = 4 \qquad C_1 = 4 - C_2$

$y = (4 - C_2)e^{t/2} + C_2 e^{-t} - te^{-t} - 2e^{-t/2}$

as $t \to \infty$ $e^{t/2}$ goes to $\infty$ $\Rightarrow$ $(4 - C_2)$ must equal $0$

$\Rightarrow C_2 = 4 \quad C_1 = 0$

$y = 4e^{-t} - te^{-t} - 2e^{-t/2}$

$= \boxed{(4-t)e^{-t} - 2e^{-t/2}}$ $\boxed{\text{Yes, unique}}$

8a) $t^2 y'' - 3ty' + 3y = 12t^4$

$t^2(\lambda)(\lambda-1)t^{\lambda-2} - 3t(\lambda)(t^{\lambda-1}) + 3t^{\lambda} = 0$

$(\lambda)(\lambda-1)t^{\lambda} - 3\lambda t^{\lambda} + 3t^{\lambda} = 0$

$(\lambda^2 - \lambda - 3\lambda + 3)t^{\lambda} = 0$

$(\lambda-1)(\lambda-3)t^{\lambda} = 0 \qquad \boxed{\lambda = 1, 3}$

7 **15 / 15**

✓ **+ 15 pts** Correct

**+ 3 pts** Found characteristic polynomial $$p(\lambda)=\lambda^2 +\frac{\lambda}{2}-\frac{1}{2}= (\lambda -\frac{1}{2})(\lambda +1)$$ for the homogeneous equation, and used it to find the general solution $$y(1)=C_1 e^{\frac{1}{2} t} + C_2 e^{-t}$$ to the homogeneous equation

**+ 8 pts** Used undetermined coefficients to solve $$2 y'' +y' -y =3 e^{-t}$$ and $$2 y'' +y' -y =2 e^{-\frac{1}{2} t}$$, getting particular solutions $$y(t)=-t e^{-t}$$ and $$y(t)=-2 e^{-\frac{1}{2} t}$$, to get the general solution $$y(t) = C_1 e^{\frac{1}{2} t} + C_2 e^{-t} -t e^{-t}-2 e^{-\frac{1}{2} t} $$

**+ 8 pts** Alternatively, used variation of parameters with $$y_1 =e^{\frac{1}{2}t}$$, $$y_2 =e^{-t}$$, $$g=\frac{3}{2} e^{-t}+e^{-\frac{1}{2}t}$$ to get a particular solution $$y_p =-\frac{2}{3} e^{-t}-2e^{-\frac{1}{2} t}-te^{-t}$$ and thence the general solution $$y(t) = C_1 e^{\frac{1}{2} t} + C_2 e^{-t} -\frac{2}{3} e^{-t}-2e^{-\frac{1}{2} t}-te^{-t}$$

**+ 4 pts** Found that the desired solution is unique $$y=4e^{-t} -2e^{-\frac{1}{2} t} -te^{-t}$$

7a) $2y''+y'-y=3e^{-t}+2e^{-t/2}$

$p(\lambda)=2\lambda^2+\lambda-1$

$=(2\lambda-1)(\lambda+1) \quad \lambda=\frac{1}{2},-1$

$g(t)=3e^{-t}+2e^{-t/2}$

$y_p(t)=Ate^{-t}+Be^{-t/2}$, since $e^{-t}$ is already a solution

$y_p'(t)=-Ate^{-t}+Ae^{-t}-\frac{1}{2}Be^{-t/2}$

$y_p''(t)=Ate^{-t}-2Ae^{-t}+\frac{1}{4}Be^{-t/2}$

$2y_p''(t)+y_p'(t)-y_p(t)=2Ate^{-t}-4Ae^{-t}+\frac{1}{2}Be^{-t/2}-Ate^{-t}-\frac{1}{2}Be^{-t/2}-Ate^{-t}-Be^{-t/2}$

$=-3Ae^{-t}-Be^{-t/2}$

$-3Ae^{-t}-Be^{-t/2}=3e^{-t}+2e^{-t/2}$

$A=-1,\ B=-2$

$y_p(t)=-te^{-t}-2e^{-t/2}$

$y=\boxed{C_1e^{t/2}+C_2e^{-t}-te^{-t}-2e^{-t/2}}$

7b) $y(0)=2=C_1+C_2-2$

$C_1+C_2=4 \qquad C_1=4-C_2$

$y=(4-C_2)e^{t/2}+C_2e^{-t}-te^{-t}-2e^{-t/2}$

as $t\to\infty$ $e^{t/2}$ goes to $\infty$ $\Rightarrow$ $(4-C_2)$ must equal 0

$\Rightarrow C_2=4 \quad C_1=0$

$y=4e^{-t}-te^{-t}-2e^{-t/2}$

$=\boxed{(4-t)e^{-t}-2e^{-t/2}}$ $\boxed{\text{Yes, unique}}$

8a) $t^2y''-3ty'+3y=12t^4$

$t^2(\lambda)(\lambda-1)t^{\lambda-2}-3t(\lambda)(t^{\lambda-1})+3t^\lambda=0$

$(\lambda)(\lambda-1)t^\lambda-3\lambda t^\lambda+3t^\lambda=0$

$(\lambda^2-\lambda-3\lambda+3)t^\lambda=0$

$(\lambda-1)(\lambda-3)t^\lambda=0 \qquad \boxed{\lambda=1,3}$

8b) $y = C_1 t^3 + C_2 t$

$y_1(t) = t^3 \quad y_2(t) = t \quad g(t) = 12t^2$

$W(t) = \det \begin{pmatrix} t^3 & t \\ 3t^2 & 1 \end{pmatrix}$

$= t^3 - 3t^3 = -2t^3$

$\int \frac{-y_2(t)g(t)}{W(t)} dt = \int \frac{-t(12t^2)}{-2t^3} = \int 6 = 6t + C$

$\int \frac{-y_1(t)g(t)}{W(t)} dt = \int \frac{t^3(12t^2)}{-2t^3} = \int -6t^2 = -2t^3 + C$

$y_p(t) = 6t^4 - 2t^4 = \boxed{4t^4}$

8c) $\boxed{y = C_1 t^3 + C_2 t + 4t^4}$

8 **12 / 12**

✓ **+ 12 pts** Correct

**+ 4 pts** Substituted $$y(t)=t^\lambda$$ in the homogeneous equation to get $$\lambda^2 -4\lambda +3=0$$, so $$\lambda \in \{ 1,3 \}$$.

**+ 4.8 pts** Remember that you need to divide by $$t^2$$, so that the coefficient of $$y''$$ is 1 and $$g(t)=12t^2$$. A common mistake was to forget to do this, giving something like $$y_p =\frac{4}{5}t^6$$ instead of the correct answer $$y_p =4t^4$$

**+ 2 pts** Added the particular solution to the general solution to the homogeneous equation (to get $$y=4 t^4 +C_1 t + C_2 t^3$$, if everything else was right)

**+ 6 pts** Took $$y_1 (t)=t , y_2 (t)=t^3$$; the Wronskian is $$W=t\cdot (3t^2) -t^3 \cdot 1 = 2t^3$$, so $$v_1 = \int \frac{-12t^5}{2t^3} dt =-2t^3$$, $$v_2 =\int \frac{12 t^3}{2 t^3} dt =6t$$, giving a particular solution $$y = v_1 y_1 + v_2 y_2 =4t^4$$