21W-MATH33B-1 Final Exam

NICHOLAS NHIEN

TOTAL POINTS

95.8 / 100

QUESTION 1

1 9 / 10

- **- 0 pts** Correct
- ✓ **- 1 pts** **Click here to replace this description.**
- **- 5 pts** Click here to replace this description.
- **- 10 pts** Click here to replace this description.
- **- 2 pts** Click here to replace this description.
- **- 3 pts** Click here to replace this description.

{ t^(n+1) / (n+1) + C } / (e^(at))

QUESTION 2

2 12 / 12

- ✓ **- 0 pts** **Correct**
- **- 1 pts** Click here to replace this description.
- **- 2 pts** Click here to replace this description.
- **- 12 pts** Click here to replace this description.

QUESTION 3

3 12 / 12

- ✓ **- 0 pts** **Correct**
- **- 2 pts** Find characteristic polynomial
- **- 2 pts** Find eigenvalues
- **- 2 pts** Find eigenvectors
- **- 2 pts** Find the general solution
- **- 1 pts** Minor error

QUESTION 4

4 15 / 15

- ✓ **- 0 pts** **Correct**
- **- 2 pts** Identify equilibria
- **- 2 pts** Correct phase line
- **- 2 pts** Correct stability
- **- 2 pts** Good plot
- **- 2 pts** Correct limits
- **- 1 pts** Minor issue

QUESTION 5

5 10 / 12

- **+ 12 pts** Correct
- ✓ **+ 10 pts** **Minor error**
- **+ 10 pts** Equation not in correct form
- **+ 0 pts** Incorrect

It looks like you missed the term with y^4 towards the end.

QUESTION 6

6 12 / 12

- ✓ **+ 12 pts** **Correct**
- **+ 3 pts** Part a
- **+ 3 pts** Part b
- **+ 3 pts** Part c
- **+ 3 pts** Part d
- **+ 0 pts** Incorrect

QUESTION 7

7 15 / 15

- ✓ **+ 15 pts** **Correct**
- **+ 3 pts** Found characteristic polynomial $$p(\lambda)=\lambda^2 +\frac{\lambda}{2}-\frac{1}{2}=(\lambda -\frac{1}{2})(\lambda +1)$$ for the homogeneous equation, and used it to find the general solution $$y(1)=C_1 e^{\frac{1}{2} t} + C_2 e^{-t}$$ to the homogeneous equation
- **+ 8 pts** Used undetermined coefficients to solve $$2 y'' +y' -y =3 e^{-t}$$ and $$2 y'' +y' -y =2 e^{-\frac{1}{2} t}$$, getting particular solutions $$y(t)=-t e^{-t}$$ and $$y(t)=-2 e^{-\frac{1}{2} t}$$, to get the general solution $$y(t) = C_1 e^{\frac{1}{2} t} + C_2 e^{-t} -t e^{-t}-2 e^{-\frac{1}{2} t} $$
- **+ 8 pts** Alternatively, used variation of parameters with $$y_1 =e^{\frac{1}{2}t}$$, $$y_2 =e^{-t}$$,

$$g=\frac{3}{2} e^{-t}+e^{-\frac{1}{2}t}$$ to get a particular solution $$y_p =-\frac{2}{3} e^{-t}-2e^{-\frac{1}{2} t}-te^{-t}$$ and thence the general solution $$y(t) = C_1 e^{\frac{1}{2} t} + C_2 e^{-t} -\frac{2}{3} e^{-t}-2e^{-\frac{1}{2} t}-te^{-t}$$

+ **4 pts** Found that the desired solution is unique $$y=4e^{-t} -2e^{-\frac{1}{2} t} -te^{-t}$$

QUESTION 8

8 10.8 / 12

+ **12 pts** Correct

✓ **+ 4 pts Substituted $$y(t)=t^\lambda$$ in the homogeneous equation to get $$\lambda^2 -4\lambda +3=0$$, so $$\lambda \in \{ 1,3 \}$$.**

✓ **+ 4.8 pts Remember that you need to divide by $$t^2$$, so that the coefficient of $$y''$$ is 1 and $$g(t)=12t^2$$. A common mistake was to forget to do this, giving something like $$y_p =\frac{4}{5}t^6$$ instead of the correct answer $$y_p =4t^4$$**

✓ **+ 2 pts Added the particular solution to the general solution to the homogeneous equation (to get $$y=4 t^4 +C_1 t + C_2 t^3$$, if everything else was right)**

+ **6 pts** Took $$y_1 (t)=t , y_2 (t)=t^3$$; the Wronskian is $$W=t\cdot (3t^2) -t^3 \cdot 1 = 2t^3$$, so $$v_1 = \int \frac{-12t^5}{2t^3} dt =-2t^3$$, $$v_2 =\int \frac{12 t^3}{2 t^3} dt =6t$$, giving a particular solution $$y = v_1 y_1 + v_2 y_2 =4t^4$$

1) $y' + ay = t^n e^{-at}$

$\mu(t) = e^{\int a\,dt}$

$= e^{at}$

$(e^{at} y)' = e^{at} t^n e^{-at}$

$= e^{at-at} t^n$

$= e t^n$

$e^{at} y = \dfrac{e t^{n+1}}{n+1}$

$y = \dfrac{e t^{n+1}}{e^{at}(n+1)}$

$$\boxed{y = \dfrac{t^{n+1}}{e^{at-1}(n+1)}}$$

2) $y^{(3)} - 4y^{(2)} - 7y' + 10y = 0$

a) $x_1 = y$ $\qquad y^{(3)} = 4y'' + 7y' - 10y$

$x_2 = x_1' = y'$

$x_3 = x_2' = x_1'' = y''$

$x_4 = x_3' = y'''$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & 7 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

b) $\det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -10 & 7 & 4-\lambda \end{pmatrix} \Rightarrow -10 - 7(-\lambda) + (4-\lambda)(\lambda^2) = 0$

$= -10 + 7\lambda + 4\lambda^2 - \lambda^3 = 0$

$= -(\lambda^3 - 4\lambda^2 - 7\lambda + 10) = 0$

$= -((\lambda-1)(\lambda+2)(\lambda-5)) = 0$

$(1-\lambda)(-2-\lambda)(5-\lambda) \Rightarrow$ Eigenvalues: $\lambda = 1, -2, 5$

$(-2-\lambda+2\lambda+\lambda^2)(5-\lambda)$

$(-2+\lambda+\lambda^2)(5-\lambda)$

$-10 + 2\lambda + 5\lambda - \lambda^2 + 5\lambda^2 - \lambda^3$

$= -10 + 7\lambda + 4\lambda^2 - \lambda^3$ ✓

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1 **9 / 10**

- **- 0 pts** Correct
- ✓ **- 1 pts** **Click here to replace this description.**
- **- 5 pts** Click here to replace this description.
- **- 10 pts** Click here to replace this description.
- **- 2 pts** Click here to replace this description.
- **- 3 pts** Click here to replace this description.

{ t^(n+1) / (n+1) + C } / (e^(at))

1) $y' + ay = t^n e^{-at}$

$\mu(t) = e^{\int a\, dt}$
$= e^{at}$

$(e^{at} y)' = e^{at} t^n e^{-at}$
$= e^{at-at} t^n$
$= e t^n$

$e^{at} y = \frac{e t^{n+1}}{n+1}$

$y = \frac{e t^{n+1}}{e^{at}(n+1)}$

$$\boxed{y = \frac{t^{n+1}}{e^{at-1}(n+1)}}$$

2) $y^{(3)} - 4y^{(2)} - 7y' + 10y = 0$

a) $x_1 = y$ $\qquad\qquad y^{(3)} = 4y'' + 7y' - 10y$
$x_2 = x_1' = y'$
$x_3 = x_2' = x_1'' = y''$
$x_4 = x_3' = y'''$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & 7 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

b) $\det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -10 & 7 & 4-\lambda \end{pmatrix} \Rightarrow -10 - 7(-\lambda) + (4-\lambda)(\lambda^2) = 0$
$= -10 + 7\lambda + 4\lambda^2 - \lambda^3 = 0$
$= -(\lambda^3 - 4\lambda^2 - 7\lambda + 10) = 0$
$= -((\lambda-1)(\lambda+2)(\lambda-5)) = 0$
$(1-\lambda)(-2-\lambda)(5-\lambda) \Rightarrow$ Eigenvalues: $\lambda = 1, -2, 5$
$(-2-\lambda+2\lambda+\lambda^2)(5-\lambda)$
$(-2+\lambda+\lambda^2)(5-\lambda)$
$-10 + 2\lambda + 5\lambda - \lambda^2 + 5\lambda^2 - \lambda^3$ $\qquad$ (2) Cont'd next page
$= -10 + 7\lambda + 4\lambda^2 - \lambda^3$ ✓

2) b) $\lambda = 1$:

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -10 & 7 & 3 & 0 \end{array}\right)$$

$$\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 10 & -7 & -3 & 0 \end{array}$$

$$\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}$$

$$\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{array}$$

$$\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \Rightarrow \begin{array}{l} x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3 \end{array} \Rightarrow \text{Eigenvec: } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = 5$:

$$\begin{array}{ccc} 5 & -1 & 0 \\ 0 & 5 & -1 \\ 10 & -7 & 1 \end{array}$$

$$\begin{array}{ccc} 5 & -1 & 0 \\ 0 & 5 & -1 \\ 0 & -5 & 1 \end{array}$$

$$\begin{array}{ccc} 5 & 0 & -\frac{1}{5} \\ 0 & 5 & -1 \\ 0 & 0 & 0 \end{array}$$

$$\begin{array}{ccc} 1 & 0 & \frac{1}{25} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 \end{array} \Rightarrow \begin{array}{l} x_1 = -\frac{x_3}{25} \\ x_2 = \frac{x_3}{5} \\ x_3 = x_3 \end{array} \Rightarrow \text{Eigenvec: } \begin{pmatrix} -\frac{1}{25} \\ \frac{1}{5} \\ 1 \end{pmatrix}$$

$\lambda = -2$:

$$+5\left(\begin{array}{ccc} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -10 & 7 & 6 \end{array}\right)$$

$$\begin{array}{ccc} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 12 & 6 \end{array}$$

$$\begin{array}{ccc} 2 & 0 & -\frac{1}{2} \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{array}$$

$$\begin{array}{ccc} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{array} \Rightarrow \begin{array}{l} x_1 = .25x_3 \\ x_2 = -.2x_3 \\ x_3 = x_3 \end{array} \Rightarrow \text{Eigenvec: } \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{5} \\ 1 \end{pmatrix}$$

c) $$x(t; C_1, C_2, C_3) = C_1 e^{t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} \frac{1}{4} \\ -\frac{1}{5} \\ 1 \end{pmatrix} + C_3 e^{5t}\begin{pmatrix} -\frac{1}{25} \\ \frac{1}{5} \\ 1 \end{pmatrix}$$

Gen sol. for $y^{(3)} - 4y^{(2)} - 7y' + 10y = 0$

$$\boxed{y(t; C_1, C_2, C_3) = C_1 e^{t} + \frac{C_2}{4} e^{-2t} - \frac{C_3}{25} e^{5t}}$$

2 12 / 12

- ✓ **- 0 pts** Correct
- **- 1 pts** Click here to replace this description.
- **- 2 pts** Click here to replace this description.
- **- 12 pts** Click here to replace this description.

3) $x' = Ax$; $A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$

$$\det\begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{pmatrix}$$

$$= (1-\lambda)(-3-\lambda) + 5$$
$$= -3 - \lambda + 3\lambda + \lambda^2 + 5$$
$$= \lambda^2 + 2\lambda + 2$$

$$\frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2}$$
$$= \frac{-2 \pm 2i}{2}$$
$$\lambda = -1 \pm i$$

$\lambda_1 = i - 1$

$$\begin{array}{cc} 1-(i-1) & -5 \\ 1 & -3-(i-1) \\ \hline 2-i & -5 \\ 1 & -2-i \\ \hline 1 & -2-i \\ 2-i & -5 \\ \hline 1 & -2-i \\ 0 & 0 \end{array}$$

$(-2-i)(2-i) = -5$

$x_1 - (2+i)x_2 = 0$

$x_1 = (2+i)x_2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$\lambda_2 = -1 - i$

$$\begin{array}{cc} 1-(-1-i) & -5 \\ 1 & -3-(-1-i) \\ \hline 2+i & -5 \\ 1 & -2+i \\ \hline 1 & i-2 \\ 2+i & -5 \\ \hline 1 & i-2 \\ 0 & 0 \end{array}$$

$(i-2)(2+i) = -5$

$x_1 + (i-2)x_2 = 0$

$x_1 = -(i-2)x_2$

$= (2-i)x_2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$$

3) $$x(t; c_1, c_2) = c_1 e^{(i-1)t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} + c_2 e^{(-i-1)t}\begin{pmatrix} 2-i \\ 1 \end{pmatrix}$$

if $\mathbb{R}$ form desired

$$e^{(i-1)t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \qquad e^{t}(e^{ti})$$

$$= e^{-t}(\cos(t) + i\sin(t))\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

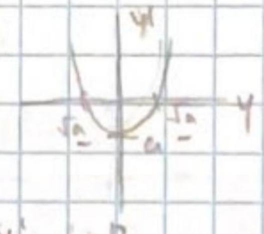
$$= e^{-t}\left(\cos t\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sin t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + ie^{-t}\left(\cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(t)\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$$

$$= e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + ie^{-t}\begin{pmatrix} \cos t + 2\sin(t) \\ \sin t \end{pmatrix}$$

$$x(t; c_1, c_2) = c_1 e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

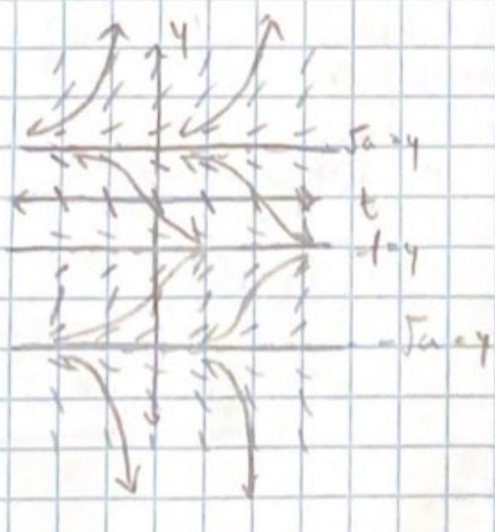
4) $y' = (y+1)(y^2 - a)$; $a \in \mathbb{R}$ & $a > 1$

$y = -1$ $\quad y^2 - a = 0$ $\quad y = \pm\sqrt{a}$



a)


y	y'
$(-\infty, \sqrt{a})$	$+ \cdot + = -$
$(\sqrt{a}, -1)$	$- \cdot - = +$
$(-1, \sqrt{a})$	$+ \cdot - = -$
$(\sqrt{a}, \infty)$	$+ \cdot + = +$

b)


$t \in (-\sqrt{a}, -1) \Rightarrow (y^2 - a) < 1$

$t \in (-1, \sqrt{a}) \Rightarrow (y^2 - a) < 1$

c)

y_0	$\lim_{t\to\infty} y(t)$
$(\sqrt{a}, \infty)$	∞
$\sqrt{a}$	$\sqrt{a}$
$(-1, \sqrt{a})$	-1
-1	-1
$(-\sqrt{a}, -1)$	-1
$-\sqrt{a}$	$-\sqrt{a}$
$(-\infty, -\sqrt{a})$	$-\infty$

3 **12** / **12**

- ✓ **- 0 pts** Correct
- **- 2 pts** Find characteristic polynomial
- **- 2 pts** Find eigenvalues
- **- 2 pts** Find eigenvectors
- **- 2 pts** Find the general solution
- **- 1 pts** Minor error

3) $$\boxed{x(t; c_1, c_2) = c_1 e^{(i-1)t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} + c_2 e^{(-i-1)t}\begin{pmatrix} 2-i \\ 1 \end{pmatrix}}$$

if $\mathbb{R}$ func desired

$e^{(i-1)t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix}$ $\quad e^{-t}(e^{ti})$

$= e^{-t}(\cos(t) + i\sin(t))\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

$= e^{-t}\left(\cos t\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sin t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + ie^{-t}\left(\cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(t)\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$

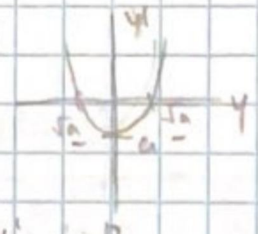
$= e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + ie^{-t}\begin{pmatrix} \cos t + 2\sin(t) \\ \sin t \end{pmatrix}$

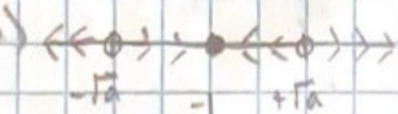
$$\boxed{x(t; c_1, c_2) = c_1 e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}}$$

4) $y' = (y+1)(y^2 - a)$; $a \in \mathbb{R}$ & $a > 1$

$y^2 - 1 \quad y^2 - a = 0$

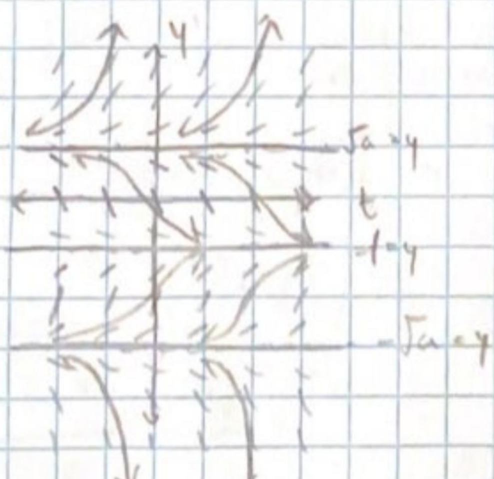
$y = \sqrt{a}$



a) 

unstable stable unstable

y	y'
$(-\infty, \sqrt{a})$	$- \cdot + = -$
$(\sqrt{a}, -1)$	$- \cdot - = +$
$(-1, \sqrt{a})$	$+ \cdot - = -$
$(\sqrt{a}, \infty)$	$+ \cdot + = +$

b) 

$t \in (-\sqrt{a}, -1) \Rightarrow (y^2 - a) < 1$

$t \in (-1, \sqrt{a}) \Rightarrow (y^2 - a) < 1$

c)

y_0	$\lim_{t\to\infty} y(t)$
$(\sqrt{a}, \infty)$	∞
$\sqrt{a}$	$\sqrt{a}$
$(-1, \sqrt{a})$	-1
-1	-1
$(-\sqrt{a}, -1)$	-1
$-\sqrt{a}$	$-\sqrt{a}$
$(-\infty, -\sqrt{a})$	$-\infty$

4 15 / 15

- ✓ **- 0 pts** **Correct**
- **- 2 pts** Identify equilibria
- **- 2 pts** Correct phase line
- **- 2 pts** Correct stability
- **- 2 pts** Good plot
- **- 2 pts** Correct limits
- **- 1 pts** Minor issue

5) $(2t^3 - 6t^2y + 3ty^2)dt + (-2t^3 + kt^2y - y^3)dy = 0$

$P = (2t^3 - 6t^2y + 3ty^2)$
$Q = (-2t^3 - kt^2y - y^3)$

$\frac{\partial P}{\partial y} = -6t^2 + 6ty$

$\frac{\partial Q}{\partial t} = -6t^2 - 2kty$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial t} \iff k = -3$

$F(t,y) = \int (2t^3 - 6t^2y + 3ty^2)dt + \int \left(-2t^3 - 3t^2y - y^3 - \frac{\partial}{\partial y}\int (2t^3 - 6t^2y + 3ty^2)dt\right)dy + C$

$= \left(\frac{1}{2}t^4 - 2t^3y + \frac{3}{2}t^2y^2\right) + \int \left(Q - \frac{\partial}{\partial y}\left(\frac{t^4}{2} - 2t^3y + \frac{3}{2}t^2y^2\right)\right)dy + C$

$+ \int Q - (-2t^3 + 3t^2y)$

$+ \int (-2t^3 - 3t^2y - y^3 + 2t^3 - 3t^2y)\,dy + C$

$+ \int (-6t^2y - y^3)\,dy$

$= \left(t^4/2 - 2t^3y + \frac{3t^2y^2}{2}\right) + \left(-3t^2y^2 - \frac{1}{4}y^4\right) + C$

$\boxed{C = \frac{t^4}{4} - \frac{3t^2y^2}{2} - 2t^3y}$

6) a) $A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$ $\det\begin{pmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{pmatrix}$

b) $\boxed{\text{No}}$

$(1-\lambda)(-7-\lambda) + 16$
$-7 - \lambda + 7\lambda + \lambda^2 + 16$
$\lambda^2 + 6\lambda + 9$
$(\lambda+3)^2$, $\boxed{\text{Eigenvalue: } \lambda = -3}$

$\lambda = 3$:

4	-4
4	-4
1	-1
0	0

$x_1 = x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x_2 = x_1$

$\boxed{\text{Eigenspace Span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

5 10 / 12

- \+ **12 pts** Correct
- ✓ + **10 pts** **Minor error**
- \+ **10 pts** Equation not in correct form
- \+ **0 pts** Incorrect

It looks like you missed the term with y^4 towards the end.

5) $(2t^3-6t^2y+3ty^2)dt+(-2t^3+kt^2y-y^3)dy=0$

$P=(2t^3-6t^2y+3ty^2)$
$Q=(-2t^3-kt^2y-y^3)$

$\frac{\partial P}{\partial y}=-6t^2+6ty$

$\frac{\partial Q}{\partial t}=-6t^2-2kty$

$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial t} \Longleftrightarrow k=-3$

$F(t,y)=\int(2t^3-6t^2y+3ty^2)dt+\int\left(-2t^3-3t^2y-y^3-\frac{\partial}{\partial y}\int(2t^3-6t^2y+3ty^2)dt\right)dy+C$

$=\left(\frac{1}{2}t^4-2t^3y+\frac{3}{2}t^2y^2\right)+\int\left(Q-\frac{\partial}{\partial y}\left(\frac{t^4}{2}-2t^3y+\frac{3}{2}t^2y^2\right)\right)dy+C$

$+\int Q-(-2t^3+3t^2y)$

$+\int(-2t^3-3t^2y-y^3+2t^3-3t^2y)dy+C$

$+\int(-6t^2y-y^3)dy$

$=\left(t^4/2-2t^3y+\frac{3t^2y^2}{2}\right)+\left(-3t^2y^2-\frac{1}{4}y^4\right)+C$

$\boxed{C=\frac{t^4}{4}-\frac{3t^2y^2}{2}-2t^3y}$

6) a) $A=\begin{pmatrix}1 & -4\\ 4 & -7\end{pmatrix}$ $\det\begin{pmatrix}1-\lambda & -4\\ 4 & -7-\lambda\end{pmatrix}$

$(1-\lambda)(-7-\lambda)+16$
$-7-\lambda+7\lambda+\lambda^2+16$
$\lambda^2+6\lambda+9$
$(\lambda+3)^2$, $\boxed{\text{Eigenvalue: } \lambda=-3}$

$\lambda+3$:

4	-4
4	-4
1	-1
0	0

$x_1=x_2 \Rightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix}=t\begin{pmatrix}1\\1\end{pmatrix}$

$x_2=x_1$

$\boxed{\text{Eigenspace Span}\begin{pmatrix}1\\1\end{pmatrix}}$

b) $\boxed{\text{No}}$

6) c) $x' = Ax$

Find second vector

4	-4	1
4	-4	1
4	-4	1
0	0	0
1	-1	1/4
0	0	0

$x_1 - x_2 = \frac{1}{4}$

$x_2 = x_2$

$x_1 = x_2 + \frac{1}{4}$

$x_2 = x_2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$

$v_2 = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$

$x_2(t) = e^{-3t}\left(\begin{pmatrix} 1/4 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

$$\boxed{x(t; C_1, C_2) = C_1 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t}\begin{pmatrix} \frac{1+4t}{4} \\ t \end{pmatrix}}$$

d) $x(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\quad \begin{pmatrix} C_1 e^{-3t} + C_2 e^{-3t}\left(\frac{1+4t}{4}\right) \\ C_1 e^{-3t} + C_2 e^{-3t} t \end{pmatrix}$

$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} C_1 e^0 + C_2 e^0 \left(\frac{1+4(0)}{4}\right) \\ C_1 e^0 + C_2 e^0 (0) \end{pmatrix}$

$= \begin{pmatrix} C_1 + \frac{1}{4}C_2 \\ C_1 \end{pmatrix}$

$C_1 = 2$; $\quad 3 = 2 + \frac{1}{4}C_2$

$1 = \frac{1}{4}C_2$

$4 = C_2$

$$\boxed{x(t) = \begin{pmatrix} 2e^{-3t} + 4e^{-3t}\left(\frac{1+4t}{4}\right) \\ 2e^{-3t} + 4e^{-3t} t \end{pmatrix}}$$

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- ✓ **+ 12 pts** Correct
- **+ 3 pts** Part a
- **+ 3 pts** Part b
- **+ 3 pts** Part c
- **+ 3 pts** Part d
- **+ 0 pts** Incorrect

7) $2y'' + y' - y = 3e^{-t} + 2e^{-t/2}$

$2y'' + y' - y = 0$

$Y_1 = 3e^{-t}$

$Y_1' = -3e^{-t}$

$Y_1'' = 3e^{-t}$

$2(3e^{-t}) - 3e^{-t} - 3e^{-t} = 0$ ✓

$Y_2 = 2e^{-t/2}$

$Y_2' = -e^{-t/2}$

$Y_2'' = \frac{e^{-t/2}}{2}$

$e^{-t/2} - e^{-t/2} - 2e^{-t/2} \neq 0$

$Y_2 = ae^{-t/2}$

$Y_2' = -\frac{a}{2}e^{-t/2}$

$Y_2'' = \frac{a}{4}e^{-t/2}$

$\frac{a}{2}e^{-t/2} - \frac{a}{2}e^{-t/2} - ae^{-t/2} = 2e^{-t/2}$

$a = 2$

$Y_{P_2} = 4e^{-t/2}$

$Y_{P_1} = ?$ $2y'' + y' + y = 3e^{-t}$

Fundamental Set of Solutions for $2y'' + y' - y = 0$

$y'' = -\frac{y'}{2} + \frac{y}{2}$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\det \begin{pmatrix} -\lambda & 1 \\ 1/2 & -1/2 - \lambda \end{pmatrix} = -\lambda\left(-\tfrac{1}{2} - \lambda\right) - \tfrac{1}{2} = \frac{\lambda}{2} + \lambda^2 - \frac{1}{2}$$

$1, 1, -\frac{1}{2}$ $\quad \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2}}{2}$

7) $2y'' + y' - y = 3e^{-t} + 2e^{t/2}$

Find Set of Solns for $2y'' + y' - y = 0$

$y'' + \frac{y'}{2} - \frac{y}{2} = 0$

$y'' = \frac{y}{2} - \frac{y'}{2}$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\det \begin{pmatrix} -\lambda & 1 \\ \frac{1}{2} & -\frac{1}{2}-\lambda \end{pmatrix}$$

$0 = -\lambda\left(-\frac{1}{2} - \lambda\right) - \frac{1}{2}$

$0 = \frac{\lambda}{2} + \lambda^2 - \frac{1}{2}$

$= 2\lambda^2 + \lambda - 1$

$\frac{-1 \pm \sqrt{1+8}}{4}$

$= \frac{-1 \pm 3}{4}$

$\lambda = -1$ or $\frac{1}{2}$

$\lambda = -1$

$\begin{matrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \hline 1 & 1 \\ 0 & 0 \end{matrix} \Rightarrow \begin{matrix} x_1 + x_2 = 0 \\ x_2 = x_2 \end{matrix}$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda = \frac{1}{2}$

$\begin{matrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & -1 \\ \hline \frac{1}{2} & -1 \\ \frac{1}{2} & -1 \\ \hline 1 & -2 \\ 0 & 0 \end{matrix}$

$x_1 - 2x_2 = 0$

$x_2 = x_2$

$x_1 = 2x_2$

$x_2 = x_2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$x(t; c_1, c_2) = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{t/2} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$y(t; c_1, c_2) = c_1 e^{-t} + c_2 e^{t/2}$

Fundamental Set of Solns: $\{e^{-t}, e^{t/2}\}$

7) Variation of Parameter

$$W = \det\begin{pmatrix} e^{-t} & e^{t/2} \\ -e^{-t} & \frac{1}{2}e^{t/2} \end{pmatrix}$$

$= \frac{1}{2}e^{t/2}e^{-t} + e^{-t}e^{t/2}$

$= \frac{3}{2}e^{-t/2}$

$v_1' = -\frac{2}{3}\frac{e^{t/2}}{e^{-t/2}}\left(\frac{3}{2}e^{-t} + e^{-t/2}\right)$ $\quad v_2' = \frac{2}{3}\frac{e^{-t}}{e^{-t/2}}\left(\frac{3}{2}e^{-t} + e^{-t/2}\right)$

$= -\frac{2}{3}e^{t}\left(\frac{3}{2}e^{-t} + e^{-t/2}\right)$ $\quad = \frac{2}{3}e^{-t/2}\left(\frac{3}{2}e^{-t} + e^{-t/2}\right)$

$= -1 - \frac{2}{3}e^{t/2}$ $\quad = e^{-3t/2} + \frac{2}{3}e^{-t}$

$\int v_1' dt = -\int (1 + \frac{2}{3}e^{t/2}) dt$ $\quad \int v_2' dt = \int (e^{-3/2 t} + \frac{2}{3}e^{-t}) dt$

$= -(t + \frac{4}{3}e^{t/2})$ $\quad = -\frac{2}{3}e^{-3/2 t} - \frac{2}{3}e^{-t}$

$y_p = -e^{-t}(t + \frac{4}{3}e^{t/2}) - e^{t/2}\left(\frac{2}{3}e^{-3/2 t} + \frac{2}{3}e^{-t}\right)$

$= -te^{-t} - \frac{4}{3}e^{-t/2} - \frac{2}{3}e^{-t} - \frac{2}{3}e^{-t/2}$

$= ce^{-t} - \frac{6}{3}e^{-t/2} - te^{-t}$

$= ce^{-t} - 2e^{-t/2} - te^{-t}$

$\boxed{y(t, c_1, c_2) = c_1e^{-t} + c_2e^{t/2} - 2e^{-t/2} - te^{-t}}$

b) $y(0) = 2$

$\lim_{t \to \infty} y(t) \neq \pm\infty$

$y' = c_1e^{-t} + c_2e^{t/2} + e^{-t/2} + te^{-t}$

$0 = c_2e^{-\infty} + c_2e^{\infty/2} + e^{-\infty/2} + \infty(0)$

$0 = c_2$

$2 = c_1e^0 + 0e^{t/2} - 2e^0 - (0)e^0$

$= c_1 - 2$

$4 = c_1$

$y(t) = 4e^{-t} - 2e^{-t/2} - te^{-t}$

$\boxed{y(t) = (4-t)e^{-t} - 2e^{-t/2}}$

Unique since $p(t) = \frac{1}{2}$, $q(t) = \frac{1}{2}$, $g(t) = \frac{3}{2}e^{-t} + e^{-t/2} : I \to \mathbb{R}$ are continuous

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✓ **+ 15 pts** Correct

+ 3 pts Found characteristic polynomial $$p(\lambda)=\lambda^2 +\frac{\lambda}{2}-\frac{1}{2}= (\lambda -\frac{1}{2})(\lambda +1)$$ for the homogeneous equation, and used it to find the general solution $$y(1)=C_1 e^{\frac{1}{2} t} + C_2 e^{-t}$$ to the homogeneous equation

+ 8 pts Used undetermined coefficients to solve $$2 y'' +y' -y =3 e^{-t}$$ and $$2 y'' +y' -y =2 e^{-\frac{1}{2} t}$$, getting particular solutions $$y(t)=-t e^{-t}$$ and $$y(t)=-2 e^{-\frac{1}{2} t}$$, to get the general solution $$y(t) = C_1 e^{\frac{1}{2} t} + C_2 e^{-t} -t e^{-t}-2 e^{-\frac{1}{2} t} $$

+ 8 pts Alternatively, used variation of parameters with $$y_1 =e^{\frac{1}{2}t}$$, $$y_2 =e^{-t}$$, $$g=\frac{3}{2} e^{-t}+e^{-\frac{1}{2}t}$$ to get a particular solution $$y_p =-\frac{2}{3} e^{-t}-2e^{-\frac{1}{2} t}-te^{-t}$$ and thence the general solution $$y(t) = C_1 e^{\frac{1}{2} t} + C_2 e^{-t} -\frac{2}{3} e^{-t}-2e^{-\frac{1}{2} t}-te^{-t}$$

+ 4 pts Found that the desired solution is unique $$y=4e^{-t} -2e^{-\frac{1}{2} t} -te^{-t}$$

8) a) $t^2 y'' - 3ty' + 3y = 12t^4$, $y(t) = t^\lambda$ for homog eqn

~~$y'' - \frac{3}{t}y' + \frac{3}{t^2}y = 12t^2$~~

$y' = \lambda t^{\lambda-1}$
$y'' = \lambda(\lambda-1)t^{\lambda-2}$

$t^2(\lambda(\lambda-1)t^{\lambda-2}) - 3t\lambda t^{\lambda-1} + 3t^\lambda = 0$

$\lambda(\lambda-1)t^\lambda - 3\lambda t^\lambda + 3t^\lambda = 0$
$t^\lambda(\lambda^2 - \lambda - 3\lambda + 3) = 0$
$t^\lambda(\lambda^2 - 4\lambda + 3) = 0$
$t^\lambda(\lambda-3)(\lambda-1) = 0$

$\boxed{\lambda = 3 \text{ or } 1}$

b) $y_1(t) = t^3$ L.I? $W = \det\begin{pmatrix} t^3 & t \\ 3t^2 & 1 \end{pmatrix}$
$y_2(t) = t$

$= t^3 - 3t^3$
$= -2t^3$
$\neq 0$

Fund set of sol: $\{t^3, t\}$

$y_p = t^3\int -\frac{t(12t^4)}{-2t^3}dt + t\int \frac{t^3(12t^4)}{-2t^3}dt$

$= t^3\int 6t\,dt - t\int 6t^3\,dt$

$= t^3 \cdot 3t^2 - t \cdot \frac{3}{2}t^4$

$= 3t^5 - \frac{3}{2}t^5$

$\boxed{= \frac{3}{2}t^5}$

c) $\boxed{y(t; C_1, C_2) = \frac{3}{2}t^5 + C_1 t^3 + C_2 t}$

8 10.8 / 12

+ **12 pts** Correct

✓ + **4 pts** **Substituted $$y(t)=t^\lambda$$ in the homogeneous equation to get $$\lambda^2 -4\lambda +3=0$$, so $$\lambda \in \{ 1,3 \}$$.**

✓ + **4.8 pts** **Remember that you need to divide by $$t^2$$, so that the coefficient of $$y''$$ is 1 and $$g(t)=12t^2$$. A common mistake was to forget to do this, giving something like $$y_p =\frac{4}{5}t^6$$ instead of the correct answer $$y_p =4t^4$$**

✓ + **2 pts** **Added the particular solution to the general solution to the homogeneous equation (to get $$y=4 t^4 +C_1 t + C_2 t^3$$, if everything else was right)**

+ **6 pts** Took $$y_1 (t)=t , y_2 (t)=t^3$$; the Wronskian is $$W=t\cdot (3t^2) -t^3 \cdot 1 = 2t^3$$, so $$v_1 = \int \frac{-12t^5}{2t^3} dt =-2t^3$$, $$v_2 =\int \frac{12 t^3}{2 t^3} dt =6t$$, giving a particular solution $$y = v_1 y_1 + v_2 y_2 =4t^4$$