22W-MATH-33B-LEC-2 Final Exam

MATTHEW CRISTOBAL NIEVA

TOTAL POINTS

98 / 100

QUESTION 1

1 10 / 10

✓ **- 0 pts** Correct

- 1 pts Click here to replace this description.

- 2 pts Click here to replace this description.

- 3 pts Click here to replace this description.

- 9 pts Click here to replace this description.

QUESTION 2

2 12 / 12

✓ **- 0 pts** Correct

- 1 pts Click here to replace this description.

- 12 pts Click here to replace this description.

QUESTION 3

3 12 / 12

✓ **- 0 pts** Correct

- 4 pts (b): v_1, v_2 and W are wrong

- 2 pts (c): wrong general solution form

- 4 pts (b): wrong homogenous solution

- 3 pts (b): wrong particular solution

- 2 pts (c): missing

- 4 pts (a): wrong λ

- 2 pts (c): it is not clear how you arrived at your conclusion

- 2 pts (c): wrong form of general solution

- 12 pts missing

QUESTION 4

4 10 / 12

- 0 pts Correct

- 3 pts should be $cos(3t)$ and $sin(3t)$

- 3 pts use initial condition to find the correct constant

- 2 pts your answer should all be in real form

✓ **- 2 pts** sign error

- 5 pts convert to real form

- 4 pts wrong set of fundamental solutions

- 8 pts this is vector differential equation, not a one dimensional differential equation

- 6 pts wrong eigenvalue

- 8 pts did not find the eigenvectors

QUESTION 5

5 15 / 15

Phase Line

✓ **- 0 pts** Correct

- 1 pts Incorrect equilibrium solutions

- 1 pts Phase line not labelled

- 2 pts Incorrect stability of solutions

- 2 pts Major error in flow along phase line

- 5 pts No phase line drawn

Direction Field

✓ **- 0 pts** Correct

- 1 pts Solution not drawn in each region (including constant solutions)

- 1 pts Slopes computed improperly or not drawn on field

- 2 pts Sample solutions incorrect

- 4 pts Equilibrium solutions missing and solutions incorrect

- 5 pts No direction field drawn

Limit of solutions as $t \to \infty$

✓ **- 0 pts** All Correct

- 1 pts Incorrect equilibrium limits

- 1 pts Incorrect limit when $y_0 > \sqrt{a}$

- 1 pts Incorrect limit when $-\sqrt{a} < y_0 < \sqrt{a}$

- 1 pts Incorrect limit when $y_0 < -\sqrt{a}$

- **5 pts** All limits computed incorrectly or problem incomplete.

QUESTION 6

6 12 / 12

Determine value of $$k$$ for which the form is exact.

✓ **- 0 pts** **Correct solution $$k=3$$**

- **1 pts** Minor computation error
- **3 pts** Major computation error
- **6 pts** No progress made towards solution

Solving the exact form.

✓ **- 0 pts** **Correct solution $$\frac 1 2 t^4 - 2 t^3 y + \frac 3 2 t^2y^2 - \frac 1 4 y^4 = C$$**

- **1 pts** Minor computation error
- **3 pts** Major computation error
- **6 pts** No progress made towards solution

QUESTION 7

7 12 / 12

✓ **+ 12 pts Correct:**
(a) $$\lambda = -3$$, $$E_{-3} = \left\{ C \cdot \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) ; C \in \mathbb{R} \right\} $$
(b) No
(c) $$ e^{-3t} \left(C_1 \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) + C_2 \left(t \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) + \left(\begin{smallmatrix} \frac{1}{4} \\ 0 \end{smallmatrix} \right) \right) \right) $$
(d) $$e^{-3t} \left(2 \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) + 4 \left(t \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) + \left(\begin{smallmatrix} \frac{1}{4} \\ 0 \end{smallmatrix} \right) \right) \right) = e^{-3t} \left(\begin{smallmatrix} 4t + 3 \\ 4t+2 \end{smallmatrix} \right)$$

+ **3 pts** Error in parts a and b: calculated the generalized eigenspace for $$\lambda = -3$$ instead of just the eigenspace, and concluded that an eigenbasis exists

+ **3 pts** $$\det (A-\lambda I) = (1-\lambda)(-7-\lambda)+16 = (\lambda +3)^2$$, so $$\lambda = -3$$ is the only eigenvalue.
The $$\lambda =-3$$-eigenspace is
$$E_{-3} = \textrm{ker} \left(\begin{smallmatrix} 4 & -4 \\ 4 & -4 \end{smallmatrix} \right) = \left\{ C \cdot \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) ; C \in \mathbb{R} \right\} .$$

+ **2 pts** No, there isn't an eigenbasis (there's only a _generalized_ eigenbasis)

+ **3 pts** One solution to
$$v_1 = \left(A + 3 I \right) v_2$$ for $$v_2$$ is
$$v_2 = \left(\begin{smallmatrix} \frac{1}{4} \\ 0 \end{smallmatrix} \right). $$ This gives the general solution
$$e^{-3t} \left(C_1 v_1 + C_2 \left(t v_1 +v_2 \right) \right)$$
$$= e^{-3t} \left(C_1 \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) + C_2 \left(t \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) + \left(\begin{smallmatrix} \frac{1}{4} \\ 0 \end{smallmatrix} \right) \right) \right) $$

+ **4 pts** Plugging in the initial condition gives
$$\begin{cases} 3=C_1 + \frac{1}{4} C_2 \\ 2 =C_1 , \end{cases}$$
so $$C_1 =2, C_2 =4$$ giving the particular solution
$$e^{-3t} \left(\begin{smallmatrix} 4t + 3 \\ 4t+2 \end{smallmatrix} \right)$$

+ **0 pts** No submission/can't find solution in document

QUESTION 8

8 15 / 15

✓ **+ 15 pts Correct:**
$$C_1 e^{\frac{1}{2}t} + C_2 e^{-t} -2e^{-\frac{1}{2} t} -t e^{-t}$$

+ **8 pts** We first solve the homogeneous equation
$$y'' + \frac{1}{2} y' - \frac{1}{2} y =0.$$
We factor the characteristic polynomial
$$\lambda^2 + \frac{1}{2} \lambda -\frac{1}{2} = \left(\lambda -\frac{1}{2} \right) (\lambda +1),$$
so we get the general solution

$$y(t) = C_1 e^{\frac{1}{2}t} + C_2 e^{-t}$$

+ 7 pts Using the method of undetermined coefficients, we get a particular solution

$$y_p (t) = -2e^{-\frac{1}{2} t} -t e^{-t},$$

so that the general solution to the original equation is

$$y(t) = C_1 e^{\frac{1}{2}t} + C_2 e^{-t} -2e^{-\frac{1}{2} t} -t e^{-t}$$

+ 12.5 pts Error in finding a particular solution to the inhomogeneous equation

$$2 y''+y'-y =3e^{-t}$$

+ 0 pts No submission

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Math 33B Final

1. (10 points) Find the general solution for the following differential equation:

$$y' + ay = t^n e^{-at},$$

where $a \in \mathbb{R}$ and $n \in \mathbb{N}$.

Integrating factor:

$$\mu(t)\,(y' + ay) = (\mu(t)y)'$$

$$\mu(t) = e^{\int a\,dt} = e^{at}$$

$$e^{at}(y' + ay) = t^n$$

$$\int (e^{at}y)'\,dy = \int t^n\,dt$$

$$e^{at}y = \frac{t^{n+1}}{n+1} + C$$

$$\boxed{y = \frac{t^{n+1}}{(n+1)e^{at}} + \frac{C}{e^{at}}}$$

1 10 / 10

- ✓ **- 0 pts** Correct
- **- 1 pts** Click here to replace this description.
- **- 2 pts** Click here to replace this description.
- **- 3 pts** Click here to replace this description.
- **- 9 pts** Click here to replace this description.

2. (12 points) Compute the general solution of the following equation by:

(a) Converting to an appropriate-sized linear system.

(b) Solving the linear system.

(c) Converting your answer back.

$$y''' - 2y'' - y' + 2y = 0.$$

a) $x_1(t) = y(t)$

y' $\quad x_1'(t) = x_2(t)$

y'' $\quad x_2'(t) = x_3(t)$ $\qquad y''' = -2y + y' + 2y''$

y''' $= \quad x_3'(t) = -2x_1(t) + x_2(t) + 2x_3(t)$

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

b) $p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & 1 & 2-\lambda \end{bmatrix}$

$= -\lambda \det \begin{bmatrix} -\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} - 1 \det \begin{bmatrix} 0 & 1 \\ -2 & 2-\lambda \end{bmatrix}$

$= -\lambda(-\lambda(2-\lambda) - 1) - 2$

$= -\lambda(-2\lambda + \lambda^2 - 1) - 2$

$= 2\lambda^2 - \lambda^3 + \lambda - 2 = -1(\lambda^3 - 2\lambda^2 - \lambda + 2)$

$= -1(\lambda-1)(\lambda+1)(\lambda-2)$

$\lambda_1 = 1, \ \lambda_2 = -1, \ \lambda_3 = 2$

$\lambda_1 = 1$:

$A - I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}$ III+2I

$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ I+II, III+II

$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 - x_3 = 0$ $\qquad x_1 = x_3$ $\quad$ let $t = x_3$

$x_2 - x_3 = 0$ $\qquad x_2 = x_3$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t = \text{null}(A - I)$ $\therefore$ eigenvector of $\lambda_1 = 1$: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$

$A + I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$ III+2I

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} I - II \\ \\ II - III \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - x_3 = 0$ $\quad x_1 = x_3$ $\quad$ Let $x_3 = t$

$x_2 + x_3 = 0$ $\quad x_2 = -x_3$

$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} t = \text{null}(A+I)$ $\quad$ eigvector of $\lambda_2 = -1$: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_3 = 2$

$$A - 2I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 1 & 0 \end{bmatrix} \begin{matrix} -I \\ \\ I + III \end{matrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} I + \tfrac{1}{2} II$$

$$= \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - \frac{1}{4}x_3 = 0$ $\quad x_2 - \frac{1}{2}x_3 = 0$

$x_1 = \frac{1}{4}x_3$ $\quad x_2 = \frac{1}{2}x_3$ $\quad$ Let $x_3 = t$

$\begin{bmatrix} 1/4 \\ 1/2 \\ 1 \end{bmatrix} t = \text{null}(A - 2I)$ $\quad$ eigvector $\lambda_3 = 2$: $\begin{bmatrix} 1/4 \\ 1/2 \\ 1 \end{bmatrix}$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \lambda = 1 \qquad v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \lambda = -1 \qquad v_3 = \begin{bmatrix} 1/4 \\ 1/2 \\ 1 \end{bmatrix} \quad \lambda = 2$$

general soln:

$$x(t, c_1, c_2, c_3) = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1/4 \\ 1/2 \\ 1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} c_1 e^{t} + c_2 e^{-t} + \frac{1}{4} c_3 e^{2t} \\ c_1 e^{t} - c_2 e^{-t} + \frac{1}{2} c_3 e^{2t} \\ c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t} \end{bmatrix}}$$

2c) Converting back...

$$y(t) = x_1(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t}$$

2 **12** / **12**

✓ **- 0 pts** Correct

- 1 pts Click here to replace this description.

- 12 pts Click here to replace this description.

3. (12 points) Consider the equation

$$t^2y'' - 3ty' + 3y = 12t^4, \quad \text{for } t > 0.$$

1)Find all real λ such that $y(t) = t^\lambda$ is a solution to the associated homogeneous equation.

2)Use the answer for part 1) and the variation of parameters method to find a particular solution to the original inhomogeneous equation.

3)Write down the general solution to the original inhomogeneous equation.

3.1) Homogeneous Eqn: $t^2y''-3ty'+3y=0,\ t>0$

Given $y(t)=t^\lambda$ $\quad y'(t)=\lambda t^{\lambda-1}$ $\quad y''(t)=(\lambda-1)(\lambda)t^{\lambda-2}$

Using above:

$$t^2(\lambda)(\lambda-1)t^{\lambda-2}-3t'\lambda t^{\lambda-1}+3t^\lambda=0$$
$$t^\lambda(\lambda)(\lambda-1)-3\lambda t^\lambda+3t^\lambda=0$$
$$t^\lambda[\lambda^2-\lambda-3\lambda+3]=0$$
$$t^\lambda[\lambda^2-4\lambda+3]=0$$
$$t^\lambda[(\lambda-3)(\lambda-1)]=0 \quad \boxed{\lambda=1,3}$$ to make $y(t)=t^\lambda$ a soln to the homogeneous eqn

3.2) Fundamental Set of Sols:

$y_1(t)=t \qquad y_2(t)=t^3$

$$t^2y''-3ty'+3y=12t^4$$
$$y''-\frac{3y'}{t}+\frac{3y}{t^2}=12t^2 \quad \therefore g(t)=12t^2$$

$$W(t)=\det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}=\det\begin{bmatrix} t & t^3 \\ 1 & 3t^2 \end{bmatrix}=3t^3-t^3=2t^3$$

$$V_1(t)=\int\frac{-y_2(t)g(t)dt}{W(t)}=\int\frac{-t^3(12t^2)}{2t^3}dt=\int\frac{-12t^5}{2t^3}dt=\int-6t^2dt=-2t^3$$

$$V_2(t)=\int\frac{y_1(t)g(t)dt}{W(t)}=\int\frac{t(12t^2)}{2t^3}dt=\int\frac{12t^3}{2t^3}dt=\int 6\,dt=6t$$

$$y_p(t)=y_1V_1+y_2V_2=-2t^4+6t^4=4t^4, \quad \boxed{y_p(t)=4t^4}$$

3.3) General Soln: $\boxed{y(t)=C_1y_1+C_2y_2+y_p=C_1t+C_2t^3+4t^4}$

3 **12** / **12**

✓ **- 0 pts** Correct

- 4 pts (b): v_1, v_2 and W are wrong

- 2 pts (c): wrong general solution form

- 4 pts (b): wrong homogenous solution

- 3 pts (b): wrong particular solution

- 2 pts (c): missing

- 4 pts (a): wrong λ

- 2 pts (c): it is not clear how you arrived at your conclusion

- 2 pts (c): wrong form of general solution

- 12 pts missing

4. (12 points) Solve the initial value problem (i) $\mathbf{x}' = A\mathbf{x}$ (ii) $\mathbf{x}(0) = \mathbf{x}_0$, where $A = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$.

and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$x' = Ax \qquad A = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$

$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 3 \\ -3 & -1-\lambda \end{bmatrix}$

$= (-1-\lambda)^2 + 9$

$= 1 + 2\lambda + \lambda^2 + 9 = \lambda^2 + 2\lambda + 10$

$\lambda = \dfrac{-2 \pm \sqrt{4 - 4(10)}}{2} = \dfrac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3i$

$\lambda_1 = -1+3i \qquad \lambda_2 = -1-3i$

$A - (-1+3i)I = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}$

$= \begin{bmatrix} 1 & i \\ -3 & -3i \end{bmatrix}$

$= \begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix} = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + x_2 i = 0 \qquad x_2 = t$

$x_1 = -it$

$\begin{bmatrix} -i \\ 1 \end{bmatrix} t = \mathrm{Null}(A - (-1+3i)I)$, eigenvector of $\lambda_1 = -1+3i$ $= \begin{bmatrix} -i \\ 1 \end{bmatrix}$

By Complex conjugates eigenvector of $\lambda_2 = -1-3i$ $= \begin{bmatrix} i \\ 1 \end{bmatrix}$

e^{3it}

$z_1(t) = e^{(-1+3i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} \qquad z_2(t) = e^{(-1-3i)t} \begin{bmatrix} i \\ 1 \end{bmatrix}$

Using euler's $e^{a+bi} = e^a(\cos b + i \sin b)$

$z_1(t) = e^{-t}(\cos(3t) + i\sin(3t))\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i\right)$

$= e^{-t}\left(\cos(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(3t)\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) + ie^{-t}\left(\cos(3t)\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$$= e^{-t}\underbrace{\begin{bmatrix} \sin 3t \\ \cos(3t) \end{bmatrix}}_{x(t)} + i e^{-t}\underbrace{\begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix}}_{y(t)}$$

$\longrightarrow$ linearly independent!

General Soln:

$$\vec{w}(t, c_1, c_2) = c_1 e^{-t}\begin{bmatrix} \sin 3t \\ \cos(3t) \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix}$$

IVP: $x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\quad w(0) = c_1 e^{0}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{0}\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad c_1 = 2, \quad c_2 = -3$$

$$\boxed{\text{IVP soln: } \quad x(t) = 2e^{-t}\begin{bmatrix} -\sin 3t \\ \cos(3t) \end{bmatrix} - 3e^{-t}\begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix}}$$

4 **10 / 12**

- **- 0 pts** Correct
- **- 3 pts** should be $$cos(3t)$$ and $$sin(3t)$$
- **- 3 pts** use initial condition to find the correct constant
- **- 2 pts** your answer should all be in real form
- ✓ **- 2 pts sign error**
- **- 5 pts** convert to real form
- **- 4 pts** wrong set of fundamental solutions
- **- 8 pts** this is vector differential equation, not a one dimensional differential equation
- **- 6 pts** wrong eigenvalue
- **- 8 pts** did not find the eigenvectors

5,

5. (15 points) Suppose $a \in \mathbb{R}$ is a fixed constant such that $a > 1$. Consider the following autonomous differential equation:

$$y' = (y+1)(y^2-a)$$

(a) Draw the corresponding phase line. Be sure to fully label the diagram which includes indicating the equilibrium points and whether they are asymptotically stable or unstable.

(b) Sketch the corresponding direction field. Include solution curves for all constant solutions and in each region between constant solutions (and above the largest constant solution and below the lowest constant solution) include a solution curve.

(c) Consider the initial value problem

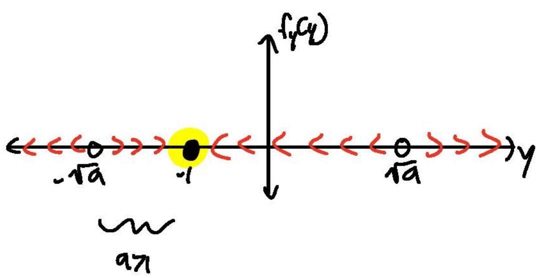
$$y' = (y+1)(y^2-a), \qquad y(0) = y_0$$

where $y_0 \in \mathbb{R}$ is an arbitrary but fixed constant. Suppose $y(t)$ is the unique solution. What is $\lim_{t\to+\infty} y(t)$? Your answer should include *all* possible cases depending on the particular value of y_0.

$y' = (y+1)(y^2-a)$, $a \in R$, $a > 1$

$= (y+1)(y+\sqrt{a})(y-\sqrt{a})$ roots: $y = -1, -\sqrt{a}, \sqrt{a}$

5a)



$(y < -\sqrt{a})$: $(-)(-)(-)$, $y' < 0$

$(-\sqrt{a} < y < -1)$: $(-)(+)(-)$, $y' > 0$

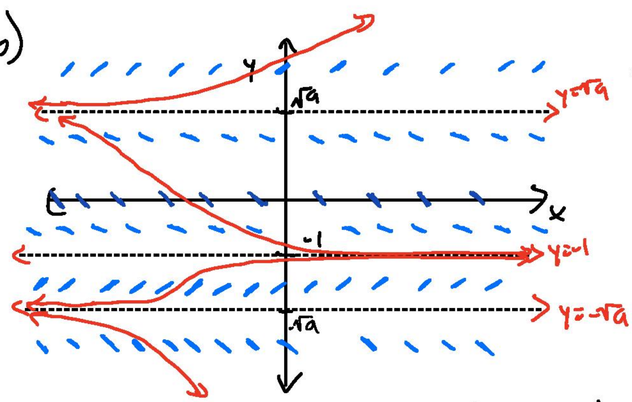
$(-1 < y < \sqrt{a})$: $(+)(+)(-)$: $y' < 0$

$(y > \sqrt{a})$: $(+)(+)(+)$: $y' > 0$

Equi points: $y=-1$ is asymptotically stable.

$y = -\sqrt{a}$ and $y = \sqrt{a}$ are asymptotically unstable

5b)



(Constant sols at dotted lines!)

$a > 1$

$y'(0) = 1(1-a)$

$y'(1) = 2(1-a)$, $(1 < a)$

$y'(2) = 3(4-a)$ $(2 > \sqrt{a}$ in this case$)$

5c) $\lim_{t\to\infty} y(t) = \begin{cases} -\infty, & y_0 < -\sqrt{a}, \\ -\sqrt{a}, & y_0 = -\sqrt{a}, \\ -1, & -\sqrt{a} < y_0 < \sqrt{a}, \\ \sqrt{a}, & y_0 = \sqrt{a}, \\ \infty, & y_0 > \sqrt{a} \end{cases}$

5 15 / 15

Phase Line

✓ **- 0 pts** **Correct**

- **- 1 pts** Incorrect equilibrium solutions
- **- 1 pts** Phase line not labelled
- **- 2 pts** Incorrect stability of solutions
- **- 2 pts** Major error in flow along phase line
- **- 5 pts** No phase line drawn

Direction Field

✓ **- 0 pts** **Correct**

- **- 1 pts** Solution not drawn in each region (including constant solutions)
- **- 1 pts** Slopes computed improperly or not drawn on field
- **- 2 pts** Sample solutions incorrect
- **- 4 pts** Equilibrium solutions missing and solutions incorrect
- **- 5 pts** No direction field drawn

Limit of solutions as $t \to \infty$

✓ **- 0 pts** **All Correct**

- **- 1 pts** Incorrect equilibrium limits
- **- 1 pts** Incorrect limit when $y_0 > \sqrt{a}$
- **- 1 pts** Incorrect limit when $-\sqrt{a} < y_0 < \sqrt{a}$
- **- 1 pts** Incorrect limit when $y_0 < -\sqrt{a}$
- **- 5 pts** All limits computed incorrectly or problem incomplete.

6. (12 points) Consider the following differential form equation

$$(2t^3 - 6t^2y + 3ty^2)dt + (-2t^3 + kt^2y - y^3)dy = 0.$$

(a) Determine the value of k such that the above equation is exact (There should be only one such value).

(b) Solve the equation for the value of k that you get in part (a).

Remark: You may leave your answer in part (b) in an implicit form.

6a) $\frac{\partial}{\partial y}(2t^3 - 6t^2y + 3ty^2) = -6t^2 + 6ty$ $\qquad \frac{\partial}{\partial t}(-2t^3 + kt^2y - y^3) = -6t^2 + 2kty$

$6ty = 2kty \qquad 2k = 6 \qquad \boxed{k=3}$

6b) $(2t^3 - 6t^2y + 3ty^2)dt + (-2t^3 + 3t^2y - y^3)dy = 0$

$F(t,y) = \int P(t,y)dt + \phi(y)$

$= \int (2t^3 - 6t^2y + 3ty^2)dt + \phi(y)$

$= \frac{1}{2}t^4 - 2t^3y + \frac{3}{2}t^2y^2 + \phi(y)$

$-2t^3 + 3t^2y + \phi'(y) = \frac{\partial}{\partial y}F(t,y) = (-2t^3 + 3t^2y - y^3)$

$\phi'(y) = -y^3 \qquad \phi(y) = -\frac{1}{4}y^4$

$\therefore F(t,y)$ is in Form:

$$\boxed{\frac{1}{2}t^4 - 2t^3y + \frac{3}{2}t^2y^2 - \frac{1}{4}y^4 = C}$$

6 **12 / 12**

Determine value of $$k$$ for which the form is exact.

✓ **- 0 pts** **Correct solution $$k=3$$**

- 1 pts Minor computation error

- 3 pts Major computation error

- 6 pts No progress made towards solution

Solving the exact form.

✓ **- 0 pts** **Correct solution $$\frac 1 2 t^4 - 2 t^3 y + \frac 3 2 t^2y^2 - \frac 1 4 y^4 = C$$**

- 1 pts Minor computation error

- 3 pts Major computation error

- 6 pts No progress made towards solution

7. (12 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}$$

(a) Find all the eigenvalues of the matrix A and for each eigenvalue determine a basis of the associated eigenspace.

(b) Does A have an eigenbasis? Yes or no (no justification needed).

(c) Determine the general solution to following linear system:

$$\mathbf{x}' = A\mathbf{x}$$

(d) Determine the specific solution to the following initial value problem:

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

7a) $p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{bmatrix}$

$= (1-\lambda)(-7-\lambda) - (4)(-4)$
$= -7 - \lambda + 7\lambda + \lambda^2 + 16$
$= \lambda^2 + 6\lambda + 9$
$= (\lambda+3)^2 \quad \boxed{\lambda_1 = \lambda_2 = -3}$

For $\lambda_1 = -3$

$A + 3I = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 - x_2 = 0$ Let $x_2 = t$
$x_1 = x_2$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} t = \text{Null}(A + 3I)$; Eigenvector: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{\text{Basis of Eigenspace: } \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}}$

7b) $\boxed{\text{No eigenbasis of A exists}}$ $p(\lambda)$ has the repeated root of $\lambda_{1,2} = -3$ which yields a singular eigenvector of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and we cannot produce another linearly independent eigenvector to complete the eigenbasis since as A is a 2x2 matrix, we need 2 linearly independent vectors to form an eigenbasis.

7c) $x' = Ax$
Using eigenvector from 7b ... $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $x_1(t) = e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
For second soln, guess

$$x(t) = e^{-3t}(V_2 + tV_1)$$

$$1) \ x(t) = e^{-3t}(Av_2 + tAv_1)$$

$$Av_2 = \lambda_1 v_2 + v_1 \qquad Av_1 = \lambda_1 v_1$$

$$(A+3I)v_2 = v_1$$

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$4x_1 - 4x_2 = 1 \qquad x_1 = x_2 + \tfrac{1}{4}$$

$$\hat{v}_2 = \begin{bmatrix} 5/4 \\ 1 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(t; C_1, C_2) = C_1 e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3t}\left(\begin{bmatrix} 5/4 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$\boxed{= \begin{bmatrix} C_1 e^{-3t} + \frac{5}{4}C_2 e^{-3t} + C_2 t e^{-3t} \\ C_1 e^{-3t} + C_2 e^{-3t} + C_2 t e^{-3t} \end{bmatrix}}$$

7d) Using $x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} C_1 + \frac{5}{4}C_2 \\ C_1 + C_2 \end{bmatrix}$$

$$3 = C_1 + \tfrac{5}{4}C_2$$

$$2 = C_1 + C_2$$

$$C_1 = 2 - C_2, \quad 3 = 2 - C_2 + \tfrac{5}{4}C_2$$

$$= 2 + \frac{C_2}{4}$$

$$1 = \frac{C_2}{4} \quad C_2 = 4$$

$$C_1 = -2, \ C_2 = 4$$

$$\boxed{x(t) = \begin{bmatrix} -2e^{-3t} + 5e^{-3t} + 4te^{-3t} \\ -2e^{-3t} + 4e^{-3t} + 4te^{-3t} \end{bmatrix}}$$

7 **12 / 12**

✓ **+ 12 pts** Correct:

(a) $$\lambda = -3$$, $$E_{-3} = \left\{ C \cdot \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) ; C \in \mathbb{R} \right\} $$

(b) No

(c) $$ e^{-3t} \left(C_1 \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) + C_2 \left(t \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) + \left(\begin{smallmatrix} \frac{1}{4} \\ 0 \end{smallmatrix} \right) \right) \right) $$

(d) $$e^{-3t} \left(2 \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) + 4 \left(t \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) + \left(\begin{smallmatrix} \frac{1}{4} \\ 0 \end{smallmatrix} \right) \right) \right) = e^{-3t} \left(\begin{smallmatrix} 4t + 3 \\ 4t+2 \end{smallmatrix} \right)$$

+ 3 pts Error in parts a and b: calculated the generalized eigenspace for $$\lambda = -3$$ instead of just the eigenspace, and concluded that an eigenbasis exists

+ 3 pts $$\det (A-\lambda I) = (1-\lambda)(-7-\lambda)+16 = (\lambda +3)^2$$, so $$\lambda = -3$$ is the only eigenvalue.
The $$\lambda =-3$$-eigenspace is
$$E_{-3} = \textrm{ker} \left(\begin{smallmatrix} 4 & -4 \\ 4 & -4 \end{smallmatrix} \right) = \left\{ C \cdot \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) ; C \in \mathbb{R} \right\} .$$

+ 2 pts No, there isn't an eigenbasis (there's only a _generalized_ eigenbasis)

+ 3 pts One solution to
$$v_1 = \left(A + 3 I \right) v_2$$ for $$v_2$$ is $$v_2 = \left(\begin{smallmatrix} \frac{1}{4} \\ 0 \end{smallmatrix} \right). $$ This gives the general solution
$$e^{-3t} \left(C_1 v_1 + C_2 \left(t v_1 +v_2 \right) \right)$$
$$= e^{-3t} \left(C_1 \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) + C_2 \left(t \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) + \left(\begin{smallmatrix} \frac{1}{4} \\ 0 \end{smallmatrix} \right) \right) \right) $$

+ 4 pts Plugging in the initial condition gives
$$\begin{cases} 3=C_1 + \frac{1}{4} C_2 \\ 2 =C_1 , \end{cases}$$
so $$C_1 =2, C_2 =4$$ giving the particular solution
$$e^{-3t} \left(\begin{smallmatrix} 4t + 3 \\ 4t+2 \end{smallmatrix} \right)$$

+ 0 pts No submission/can't find solution in document

8. (15 points) Find the general solution to the equation $y(t)$ to

$$2y'' + y' - y = 3e^{-t} + 2e^{-t/2}$$

Hint: Find general solution to the homogenous equation. Using the method of undetermined coefficients or using the variation of parameters method to find a particular solution.

Show your work/Justify your answer!

Homogeneous $2y'' + y' - y = 0$

$$y'' + \frac{y'}{2} - \frac{y}{2} = 0$$

$$f(\lambda) = \lambda^2 + \frac{\lambda}{2} - \frac{1}{2} = 0$$

$$(\lambda+1)(\lambda - \tfrac{1}{2}) = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = \tfrac{1}{2} \qquad Y_1 = e^{-t} \quad Y_2 = e^{t/2}$$

$$y(t; C_1, C_2) = C_1 e^{-t} + C_2 e^{t/2}$$

0 b doesn't work, test ate^{-t}

$$Y_p(t) = ae^{-t} + be^{-t/2}$$

$$y'_p(t) = -ae^{-t} - \tfrac{1}{2}be^{-t/2}$$

$$y''_p(t) = ae^{-t} + \tfrac{1}{4}be^{-t/2}$$

$$2ae^{-t} + \tfrac{1}{2}be^{-t/2} - ae^{-t} + \tfrac{1}{2}be^{-t/2} - ae^{-t} - be^{-t/2}$$

$$= 0 = ae^{-t} + be^{-t/2}$$

Inhomogeneous: $2y'' + y' - y = 3e^{-t} + 2e^{-t/2}$

Undetermined Coefficients

Trial S/n: $Y_p(t) = ate^{-t} + be^{-t/2}$

$$y_p'(t) = -ate^{-t} + ae^{-t} - \tfrac{1}{2}be^{-t/2}$$

$$y''_p(t) = ate^{-t} - ae^{-t} - ae^{-t} + \tfrac{1}{4}be^{-t/2}$$

$$= ate^{-t} - 2ae^{-t} + \tfrac{1}{4}be^{-t/2}$$

$$2y'' + y' - y = 3e^{-t} + 2e^{-t/2}$$

$$2\left(ate^{-t} - 2ae^{-t} + \frac{b}{4}e^{-t/2}\right) + \left(-ate^{-t} + ae^{-t} - \frac{b}{2}e^{-t/2}\right) - \left(ate^{-t} + be^{-t/2}\right)$$

$$= 3e^{-t} + 2e^{-t/2}$$

$$2a\cancel{te}^{-t} - 4ae^{-t} + \frac{b}{2}e^{-t/2} - a\cancel{te}^{-t} + ae^{-t} - \frac{b}{2}e^{-t/2} - a\cancel{te}^{-t} - be^{-t/2} = 3e^{-t} + 2e^{-t/2}$$

$$-3ae^{-t} - be^{-t/2} = 3e^{-t} + 2e^{-t/2}$$

$$a = -1, \quad b = -2 \qquad \therefore Y_p(t) = -te^{-t} - 2e^{-t/2}$$

Using our answer from homogeneous,

General Solution: $\boxed{y(t) = C_1 e^{-t} + C_2 e^{t/2} - te^{-t} - 2e^{-t/2}}$

8 **15 / 15**

✓ **+ 15 pts** Correct:

$$C_1 e^{\frac{1}{2}t} + C_2 e^{-t} -2e^{-\frac{1}{2} t} -t e^{-t}$$

+ 8 pts We first solve the homogeneous equation

$$y'' + \frac{1}{2} y' - \frac{1}{2} y =0.$$

We factor the characteristic polynomial

$$\lambda^2 + \frac{1}{2} \lambda -\frac{1}{2} = \left(\lambda -\frac{1}{2} \right) (\lambda +1),$$

so we get the general solution

$$y(t) = C_1 e^{\frac{1}{2}t} + C_2 e^{-t}$$

+ 7 pts Using the method of undetermined coefficients, we get a particular solution

$$y_p (t) = -2e^{-\frac{1}{2} t} -t e^{-t},$$

so that the general solution to the original equation is

$$y(t) = C_1 e^{\frac{1}{2}t} + C_2 e^{-t} -2e^{-\frac{1}{2} t} -t e^{-t}$$

+ 12.5 pts Error in finding a particular solution to the inhomogeneous equation

$$2 y''+y'-y =3e^{-t}$$

+ 0 pts No submission