21F-MATH33B-1 Final Exam

TOTAL POINTS

100 / 100

QUESTION 1

1 12 / 12

✓ **- 0 pts** Correct

- 1 pts Click here to replace this description.

- 9 pts Click here to replace this description.

- 4 pts Click here to replace this description.

- 2 pts Click here to replace this description.

- 3 pts Click here to replace this description.

QUESTION 2

2 16 / 16

✓ **- 0 pts** Correct

- 1 pts Click here to replace this description.

- 2 pts Click here to replace this description.

- 10 pts Click here to replace this description.

- 7 pts Click here to replace this description.

QUESTION 3

3 15 / 15

✓ **- 0 pts** Correct

- 4 pts Incorrect general solutions

- 3 pts Incorrect particular solution

- 2 pts Computational mistakes

- 2 pts Incorrect eigenvalues

- 2 pts Incorrect eigenvectors

- 2 pts Incorrect solution for initial value problem

QUESTION 4

4 10 / 10

✓ **- 0 pts** Correct

- 3 pts Incorrect partial derivatives

- 4 pts Incorrect definition of exact differential forms

QUESTION 5

5 20 / 20

Part A

✓ **+ 2 pts** Correct characteristic polynomial.

✓ **+ 2 pts** Correct eigenvalues.

Part B

✓ **+ 4 pts** Correctly found an eigenvector.

Part C

✓ **+ 4 pts** Correct generalized eigenvector.

Part D

✓ **+ 4 pts** Correct general solution.

Part E

✓ **+ 2 pts** Correctly plugged in the initial condition.

✓ **+ 2 pts** Correct answer.

+ 0 pts Incorrect & not enough correct work shown.

QUESTION 6

6 12 / 12

✓ **- 0 pts** Correct

- 4 pts Incorrect Characteristic Polynomial.

- 4 pts Incorrect Eigenvalues.

- 4 pts Incorrect General Solution.

QUESTION 7

7 15 / 15

✓ **- 0 pts** Correct

- 3 pts Incorrect integrating factor.

- 2 pts Computation/integration errors.

- 5 pts Incorrect/missing general solution to the differential equation.

- 3 pts Incorrect/missing particular solution to the IVP.

- 15 pts Incorrect/missing

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1. (12 points) Find the general solution for $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$P(\lambda) = \det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)^2$$

$\Rightarrow$ eigenvalues: $\lambda_1 = \lambda_2 = 1$, double real roots

eigenvectors:

null $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\begin{cases} x_1 = x_1 \\ x_2 = x_2 \end{cases}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2$

$\therefore$ two eigenvectors: $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and they are linearly independent

$\therefore$ general solution: $\vec{x}(t; C_1, C_2) = C_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

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2. (16 points) (a) Converting the following equation to a system of first order differential equations.

(b) Find the corresponding characteristic polynomial and all the eigenvalues.

$$y''' - 2y'' - y' + 2y = 0.$$

a) $x_1 = y$

$x_2 = x_1' = y'$

$x_3 = x_2' = y''$

$x_3' = y''' = 2y'' + y' - 2y = 2x_3 + x_2 - 2x_1$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

b) $P(\lambda) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & 1 & 2-\lambda \end{bmatrix}$ expand along this row

$= (-1)^{3+1}(-2)\det\begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix} + (-1)^{3+2}(1)\det\begin{bmatrix} -\lambda & 0 \\ 0 & 1 \end{bmatrix} + (-1)^{3+3}(2-\lambda)\det\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$

$= (-2)(1) + (-1)(-\lambda) + (2-\lambda)(-\lambda)^2$

$= -2 + \lambda + 2\lambda^2 - \lambda^3$

$= -\lambda^2(\lambda - 2) + (\lambda - 2)$

$= (-\lambda^2 + 1)(\lambda - 2)$

$= -(\lambda^2 - 1)(\lambda - 2)$

$= -(\lambda + 1)(\lambda - 1)(\lambda - 2)$

$\therefore$ eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$

$\lambda = \alpha + \beta i, \quad \vec{v}_1 = \vec{w}_1 + i\vec{w}_2$

$\vec{x}(t; C_1, C_2) = C_1 e^{\alpha t}(\cos\beta t\,\vec{w}_1 - \sin\beta t\,\vec{w}_2) + C_2 e^{\alpha t}(\sin\beta t\,\vec{w}_1 + \cos\beta t\,\vec{w}_2)$

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3. (15 points) Solve the initial value problem of the form (i) $\mathbf{x}' = A\mathbf{x}$ (ii) $\mathbf{x}(0) = \mathbf{x}_0$, where $A = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

$P(\lambda) = \det \begin{bmatrix} -1-\lambda & 3 \\ -3 & -1-\lambda \end{bmatrix}$

$= (-1-\lambda)^2 + 9$

$= (\lambda+1)^2 + 9$

$= \lambda^2 + 2\lambda + 1 + 9$

$= \lambda^2 + 2\lambda + 10$

$\lambda = \dfrac{-2 \pm \sqrt{4 - 4\times 10}}{2}$

$= \dfrac{-2 \pm \sqrt{-36}}{2}$

$= \dfrac{-2 \pm 6i}{2}$

$= -1 \pm 3i$

eigenvalues: $\lambda_1 = -1+3i$,

$\lambda_2 = -1-3i$

eigenvectors: $\lambda_1 = -1+3i$:

$\text{null}\begin{bmatrix} -1-(-1+3i) & 3 \\ -3 & -1-(-1+3i) \end{bmatrix}$

$= \text{null}\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}$

$\left[\begin{array}{cc|c} -3i & 3 & 0 \\ -3 & -3i & 0 \end{array}\right]$

$= \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right]$

$= \left[\begin{array}{cc|c} 1 & \frac{1}{-i} & 0 \\ -1 & -i & 0 \end{array}\right]$

$= \left[\begin{array}{cc|c} 1 & i & 0 \\ -1 & -i & 0 \end{array}\right]$

$= \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = -ix_2 \\ x_2 = x_2 \end{cases}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} x_2 \Rightarrow \vec{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \overline{\vec{v}_1} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda_1 = -1+3i, \quad \vec{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$

$\therefore$ general solution:

$\vec{x}(t; C_1, C_2)$

$= C_1 e^{-t}\left(\cos(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(3t)\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) +$

$C_2 e^{-t}\left(\sin(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos(3t)\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$

$\vec{x}(0)$

$= C_1\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + C_2\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\left[\begin{array}{cc|c} 0 & -1 & 3 \\ 1 & 0 & 2 \end{array}\right]$

$\therefore C_1 = 2, \ C_2 = -3$

$\therefore \vec{x}(t) = 2e^{-t}\left(\cos(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(3t)\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) +$

$(-3e^{-t})\left(\sin(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos(3t)\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$

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4. (10 points) Check whether the following differential form is exact.

$$(t^3 - t^2 \sin y - y)dy \ + \ (2t \cos y + 3t^2 y)dt$$

$P(t, y) = t^3 - t^2 \sin y - y$

$Q(t, y) = 2t \cos y + 3t^2 y$

$\frac{\partial P}{\partial t} = 3t^2 - 2t \sin y \quad , \quad \frac{\partial Q}{\partial y} = -2t \sin y + 3t^2$

$\because \frac{\partial P}{\partial t} = \frac{\partial Q}{\partial y}$

$\therefore$ the differential form is closed

$\because$ domains of P and Q are both rectangular ($\mathbb{R} \times \mathbb{R}$) and are both continuous and diffrentiable

$\therefore$ the differential form is exact

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5. (20 points) This problem covers the "double real root interesting case". We will find the general solution to the equation:

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$$

(a) Find the characteristic polynomial $p(\lambda)$ of A, then use this polynomial to determine the eigenvalues of A.

(b) In this case, $\lambda := \lambda_1 = \lambda_2$ is an eigenvalue of multiplicity two. Since this is the interesting case, there is only one linearly independent eigenvector associated to λ. Find it.

(c) Let $\mathbf{v}_1$ be the eigenvector found in step (b). Find a second vector $\mathbf{v}_2$ which is a solution to the matrix equation:

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1.$$

(d) Write down the general solution:

$$\mathbf{x}(t) = C_1 e^{\lambda t}\mathbf{v}_1 + C_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$$

(e) Solve the initial value problem:

$\mathbf{x}' = A\mathbf{x}$

$\mathbf{x}(0) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

a) $P(\lambda) = \det\begin{bmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix}$

$= (-3-\lambda)(-1-\lambda)+1$

$= 3+\lambda+3\lambda+\lambda^2+1$

$= \lambda^2+4\lambda+4$

$= (\lambda+2)^2$

$\therefore$ eigenvalue : $\lambda_1 = \lambda_2 = -2$

b) eigenvector

$\text{null}\begin{bmatrix} -3+2 & 1 \\ -1 & -1+2 \end{bmatrix}$

$= \text{null}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right]$

$= \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{cases} x_1 = x_2 \\ x_2 = x_2 \end{cases}$, $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

c) $(A-\lambda I)\vec{v_2} = \vec{v_1}$

$\begin{bmatrix} -3+2 & 1 \\ -1 & -1+2 \end{bmatrix}\vec{v_2} = \vec{v_1}$

$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right]$

$= \left[\begin{array}{cc|c} -1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$

$\begin{cases} -x_1 + x_2 = 1 \\ x_2 = x_2 \end{cases}$

$\Rightarrow \begin{cases} x_1 = x_2 - 1 \\ x_2 = x_2 \end{cases}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

let $x_2 = 0$,

$\vec{v_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\vec{v_1}$ and $\vec{v_2}$ are linearly independent

d)

$\vec{x}(t) = C_1 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-2t}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

e) $\vec{x}(0) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$\vec{x}(0) = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$\begin{bmatrix} C_1 & -C_2 \\ C_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$C_1 = -3$,

$C_1 - C_2 = 0$

$C_1 = C_2$

$C_2 = -3$

$\therefore \vec{x}(t) = -3e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3e^{-2t}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$C_1 e^{at}\cos bt + C_2 e^{at}\sin bt$

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6. (12 points) Find the general solution to the following homogeneous second-order linear differential equation:

$$y'' - 2y' + 5y = 0$$

characteristic equation:

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4 - 4\times 5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$\lambda_1 = 1 + 2i$

$Re(\lambda_1) = 1$, $Im(\lambda_1) = 2$

general solution:

$$\vec{x}(t; C_1, C_2) = C_1 e^{t}\cos 2t + C_2 e^{t}\sin 2t$$

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7. (15 points) Solve the following initial value problem.

$$y' - \frac{3}{t}y = t^2 + \frac{3}{t}, \quad y(-1) = 0.$$

Integrating factor:

$$\mu(t) = \exp\int -\frac{3}{t}\,dt$$
$$= \exp\left(-3\int \frac{1}{t}\,dt\right)$$
$$= \exp(-3\ln|t|)$$
$$= \exp(\ln t^{-3})$$
$$= t^{-3}$$
$$= \frac{1}{t^3}$$

Multiply both sides by $\mu(t)$:

$$\frac{1}{t^3}y' - \frac{3}{t^4}y = \frac{1}{t} + \frac{3}{t^4}$$

$$\int \frac{1}{t^3}y' - \frac{3}{t^4}y\,dt = \int \frac{1}{t} + \frac{3}{t^4}\,dt$$

$$\frac{1}{t^3}y = \ln|t| - t^{-3} + C$$

$$y = t^3\ln|t| - 1 + t^3C, \quad t \in (-\infty, 0) \cup (0, \infty)$$

$$y(-1) = (-1)\ln|1| - 1 + (-1)^3C$$
$$= -1 - C = 0$$
$$C = -1$$

$$\therefore y(t) = t^3\ln|t| - 1 - t^3 \qquad \because -1 \in (-\infty, 0)$$
$$\therefore t \in (-\infty, 0)$$