University of California, Los Angeles
Fall 2021

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Date: Dec 4, 2021

MATH 33B: Differential Equations
Final Exam

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1. (12 points) Find the general solution for $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(1-\lambda) = 0$$

$$\lambda = 1.$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_1 = x_1$$
$$x_2 = x_2$$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\boxed{x(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

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2. (16 points) (a) Converting the following equation to a system of first order differential equations.
   (b) Find the corresponding characteristic polynomial and all the eigenvalues.

$$y''' - 2y'' - y' + 2y = 0.$$

a)

$$\boxed{\begin{array}{c} y' = Ay \\ A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \end{array}}$$

$y_1 = y$
$y_2 = y'$
$y_3 = y''$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{pmatrix} y_2 \\ y_3 \\ 2y_3 + y_2 - 2y_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{y}$$

$$\det(A - \lambda I) = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & 1 & 2-\lambda \end{pmatrix} = -\lambda \begin{pmatrix} -\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} - 1 \begin{pmatrix} 0 & 1 \\ -2 & 2-\lambda \end{pmatrix}$$

$$(-\lambda)((-\lambda)(2-\lambda) - 1) - 2$$

b) $\boxed{P(\lambda) = -\lambda^3 + 2\lambda^2 + \lambda - 2 = 0}$

$$\begin{array}{c|cccc} 1 & -1 & 2 & 1 & -2 \\ & & -1 & 1 & 2 \\ \hline & -1 & 1 & 2 & 0 \end{array}$$

$$(\lambda - 1)(-\lambda^2 + \lambda + 2) = 0$$
$$(\lambda - 1)(\lambda + 1)(-\lambda + 2) = 0$$

b) $\boxed{\lambda = -1, 1, 2}$

$\lambda = -1$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda = 1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = 2$

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

$$y = c_1 e^{-t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{2t}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \xrightarrow{\text{take first row}} y = c_1 e^{-t} + c_2 e^{t} + c_3 e^{2t}$$

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3. (15 points) Solve the initial value problem of the form (i) $\mathbf{x}' = A\mathbf{x}$ (ii) $\mathbf{x}(0) = \mathbf{x}_0$, where $A = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{pmatrix} -1-\lambda & 3 \\ -3 & -1-\lambda \end{pmatrix} = (-1-\lambda)(-1-\lambda) + 9$$

$$1 + 2\lambda + \lambda^2 + 9$$

$$\lambda^2 + 2\lambda + 10 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm 3i$$

$\lambda = -1 + 3i$

$$\begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$x_1 = 1$

$-3i + 3x_2 = 0$ $\quad \begin{pmatrix} 1 \\ i \end{pmatrix}$

$x_2 = i$

$$X = c_1 e^{(-1+3i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{(-1-3i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$\Downarrow$

$$e^{-t} e^{3it}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$e^{-t}(\cos(3t) + i\sin(3t))\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$e^{-t}\begin{pmatrix} \cos(3t) + i\sin(3t) \\ i\cos(3t) + i^2\sin(3t) \end{pmatrix}$$

$$X = c_1 e^{-t}\begin{pmatrix} \cos(3t) \\ -\sin(3t) \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin(3t) \\ \cos(3t) \end{pmatrix}$$

$$x_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$c_1 \cdot \cos 0 = 3 \qquad c_1 = 3$

$c_2 \cos 0 = 2 \qquad c_2 = 2$

$$\boxed{x = 3e^{-t}\begin{pmatrix} \cos(3t) \\ -\sin(3t) \end{pmatrix} + 2e^{-t}\begin{pmatrix} \sin(3t) \\ \cos(3t) \end{pmatrix}}$$

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4. (10 points) Check whether the following differential form is exact.

$$(t^3 - t^2 \sin y - y)dy \ + \ (2t\cos y + 3t^2 y)dt$$

P Q

$\frac{dP}{dt} = \frac{d}{dt} t^3 - t^2 \sin y - y = 3t^2 - 2t\sin y$

$\frac{dQ}{dy} = \frac{d}{dy} 2t\cos y + 3t^2 y = -2t\sin y + 3t^2$

$\frac{dP}{dt} = \frac{dQ}{dy}$ so it is exact.

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5. (20 points) This problem covers the "double real root interesting case". We will find the general solution to the equation:

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$$

(a) Find the characteristic polynomial $p(\lambda)$ of A, then use this polynomial to determine the eigenvalues of A.

(b) In this case, $\lambda := \lambda_1 = \lambda_2$ is an eigenvalue of multiplicity two. Since this is the interesting case, there is only one linearly independent eigenvector associated to λ. Find it.

(c) Let $\mathbf{v}_1$ be the eigenvector found in step (b). Find a second vector $\mathbf{v}_2$ which is a solution to the matrix equation:

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1.$$

(d) Write down the general solution:

$$\mathbf{x}(t) = C_1 e^{\lambda t}\mathbf{v}_1 + C_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$$

(e) Solve the initial value problem:

$\mathbf{x}' = A\mathbf{x}$

$\mathbf{x}(0) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{pmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = (-3-\lambda)(-1-\lambda) + 1 = 0$$

$$3 + \lambda + 3\lambda \quad + \lambda^2 + 1$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda + 2)^2 = 0$$

$\lambda = -2$ w/ multiplicity 2

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$-x_1 + x_2 = 0$

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$x(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t}\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$x(0) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$ $\quad c_1 - c_2 = 0$

$c_1 = -3$

$c_2 = -3$

$-x_1 + x_2 = 1$

$v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$\boxed{x(t) = -3e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3e^{-2t}\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)}$$

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6. (12 points) Find the general solution to the following homogeneous second-order linear differential equation:

$$y'' - 2y' + 5y = 0$$

$p(\lambda) = \lambda^2 - 2\lambda + 5 = 0$

$\lambda = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$

$\boxed{y(t) = C_1 e^t \cos(2t) + C_2 e^t \sin(2t)}$

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7. (15 points) Solve the following initial value problem.

$$y' - \frac{3}{t}y = t^2 + \frac{3}{t}, \quad y(-1) = 0.$$

$$\mu(t) = e^{\int -\frac{3}{t}\,dt} = e^{-3\ln t} = t^{-3}$$

$$t^{-3}y' - 3t^{-4}y = t^{-1} + 3t^{-4}$$

$$\int (t^{-3}y)' = \int t^{-1} + 3t^{-4}\,dt$$

$$t^{-3}y = \ln|t| - t^{-3} + C$$

$$y = t^3\ln|t| - 1 + Ct^3$$

$$y(-1) = 0$$

$$0 = 0 - 1 - C \Rightarrow C = -1$$

$$\boxed{y(t) = t^3\ln|t| - t^3 - 1}$$

$$y' = 3t^2\ln t + t^2 - 3t^2 \quad - 3t^2\ln t + 3t^2 + \frac{3}{t}$$