University of California, Los Angeles
Fall 2021

Instructor: C. Wang
Date: Dec 4, 2021

MATH 33B: DIFFERENTIAL EQUATIONS
FINAL EXAM

Last Name Xiyang

First Name Zhou

Student ID 405571728

TA Name Thomas Brown

Last Name Zhou , First Name Yuyang ,
Student ID 405571728

1. (12 points) Find the general solution for $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Find eigenvalues: $p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = 0$

$\lambda = 1$

Find eigenvalues: $\operatorname{null}(A - I) = \operatorname{null} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{cases} x_1 = x_1 \\ x_2 = x_2 \end{cases}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

General solution: $\vec{x}(t) = C_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1, C_2 \in \mathbb{R}$

Last Name Zhou, First Name Xuyang,
Student ID 405571728

2. (16 points) (a) Converting the following equation to a system of first order differential equations.
(b) Find the corresponding characteristic polynomial and all the eigenvalues.

$$y''' - 2y'' - y' + 2y = 0.$$

(a) Let: $x_1 = y$, $x_2 = y'$, $x_3 = y''$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -2x_1 + x_2 + 2x_3 \end{cases}$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \vec{x}$$

(b) $P(\lambda) = \underset{(-1)^3}{\wedge} \det(A - \lambda I) = -\det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & 1 & 2-\lambda \end{bmatrix} = \lambda\left((\lambda)(\lambda-2) - 1\right) + 2$

$= \lambda(\lambda^2 - 2\lambda - 1) + 2$

$= \lambda^3 - 2\lambda^2 - \lambda + 2$

$P(\lambda) = 0$: $\lambda^3 - 2\lambda^2 - \lambda + 2 = \lambda(\lambda^2 - 1) - 2(\lambda^2 - 1)$

$= (\lambda + 1)(\lambda - 1)(\lambda - 2) = 0$

$\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$

Last Name Zhou, First Name Xuyang,
Student ID 405571728

3. (15 points) Solve the initial value problem of the form (i) $\mathbf{x}' = A\mathbf{x}$ (ii) $\mathbf{x}(0) = \mathbf{x}_0$, where $A = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

$$\vec{x}' = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix} \vec{x}$$

· Find eigenvalues: $p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 3 \\ -3 & -1-\lambda \end{bmatrix} = (-1-\lambda)^2 + 9 = 0$

$$(\lambda+1)^2 = -9, \quad \lambda = -1 \pm 3i$$

· Find eigenvectors: $\text{null}(A - (-1+3i)I) = \text{null} \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}$

$$\text{rref} \left[\begin{array}{cc|c} -3i & 3 & 0 \\ -3 & -3i & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\begin{cases} x_1 = -x_2 i \\ x_2 = x_2 \end{cases}$ $\Rightarrow$ eigenvectors are: $\vec{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ (because two eigenvectors are conjugate)

· General solution: $\vec{x}(t) = C_1 e^{-t} \left(\cos(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(3t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$$+ C_2 e^{-t} \left(\sin(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos(3t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

· Plug in IV: $\vec{x}(0) = C_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$C_1 = 2, \quad C_2 = 3$$

$$\vec{x}(t) = 2e^{-t}\left(\cos(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(3t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$+ 3e^{-t}\left(\sin(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos(3t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

Last Name Zhou, First Name Yuyang,
Student ID 405571728

4. (10 points) Check whether the following differential form is exact.

$$(t^3 - t^2 \sin y - y)dy + (2t\cos y + 3t^2 y)dt$$

$\frac{\partial}{\partial t}(t^3 - t^2 \sin y - y) = 3t^2 - 2t\sin y$

$\frac{\partial}{\partial y}(2t\cos y + 3t^2 y) = -2t\sin y + 3t^2 = \frac{\partial}{\partial t}(t^3 - t^2 \sin y - y)$

So, the DF is closed.

Because the domain of both $(t^3 - t^2 \sin y - y)$ and $(2t\cos y + 3t^2 y)$ are $\mathbb{R}\times\mathbb{R}$, a rectangle area on t-y plane, the DF is also exact.

Last Name Zhou, First Name Xuyang,
Student ID 405571228

5. (20 points) This problem covers the "double real root interesting case". We will find the general solution to the equation:

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$$

(a) Find the characteristic polynomial $p(\lambda)$ of A, then use this polynomial to determine the eigenvalues of A.

(b) In this case, $\lambda := \lambda_1 = \lambda_2$ is an eigenvalue of multiplicity two. Since this is the interesting case, there is only one linearly independent eigenvector associated to λ. Find it.

(c) Let $\mathbf{v}_1$ be the eigenvector found in step (b). Find a second vector $\mathbf{v}_2$ which is a solution to the matrix equation:

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1.$$

(d) Write down the general solution:

$$\mathbf{x}(t) = C_1 e^{\lambda t}\mathbf{v}_1 + C_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$$

(e) Solve the initial value problem:
$\mathbf{x}' = A\mathbf{x}$
$\mathbf{x}(0) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

(a) $p(\lambda) = \det(A-\lambda I) = \det\begin{bmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix} = (\lambda+3)(\lambda+1)+1 = \lambda^2+4\lambda+4 = (\lambda+2)^2$

$(\lambda+2)^2 = 0 \Rightarrow \lambda = -2$

(b) $\text{null}(A+2I) = \text{null}\left(\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\right)$

$\text{rref}\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$, $\begin{cases} x_1 = x_2 \\ x_2 = x_2 \end{cases}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $(A+2I)\vec{v}_2 = \vec{v}_1$

$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, take $\vec{v}_2 = \begin{bmatrix} 0 \\ a \end{bmatrix}$ to calculate. $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(d) $\vec{x}(t) = C_1 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-2t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

(e) $\vec{x}(0) = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$\begin{cases} C_1 = 0 \\ C_2 = -3 \end{cases} \Rightarrow \vec{x}(t) = -3e^{-2t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

Last Name Zhou, First Name Xnyang,
Student ID 405571728

6. (12 points) Find the general solution to the following homogeneous second-order linear differential equation:

$$y'' - 2y' + 5y = 0$$

The characteristic polynomial: $p(\lambda) = \lambda^2 - 2\lambda + 5$

$p(\lambda) = 0$ 时, $\lambda^2 - 2\lambda + 5 = 0$

$(\lambda-1)^2 + 4 = 0$

$(\lambda-1)^2 = -4$

$\lambda = 1 \pm 2i$

General solution: $y(t) = C_1 e^t \cos 2t + C_2 e^t \sin 2t$, $C_1, C_2 \in \mathbb{R}$

Last Name Zhou, First Name Xuyang,
Student ID 405571728

7. (15 points) Solve the following initial value problem.

$$y' - \frac{3}{t}y = t^2 + \frac{3}{t}, \quad y(-1) = 0.$$

$y' - \frac{3}{t}y = t^2 + \frac{3}{t}$

$\mu(t) = \exp(\int -\frac{3}{t}dt) = \exp(-3\ln|t|) = t^{-3}$

Multiply $\mu(t)$ to the equation:

$(t^{-3}y)' = \frac{1}{t} + \frac{3}{t^4}$

$t^{-3}y = \int \frac{1}{t} + \frac{3}{t^4}\,dt = \ln|t| - \frac{1}{t^3} + C$

$y = t^3\ln|t| - 1 + t^3 \cdot C$

Plug in $y(-1) = 0$:

$y(-1) = -1 - C = 0$

$C = -1$

$y(t) = t^3\ln|t| - 1 - t^3$