University of California, Los Angeles
Spring 2020

Instructor: C. Wang
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MATH 33B: Differential Equations
Final Exam

This exam contains 4 pages (including this cover page) and 8 problems. The following rules apply:

- **You have 24 hours to submit this take home exam**. **The deadline for submission is Tuesday, Dec. 15, 8:00am (PT)**. Please, note that this deadline applies to all students, including those registered with CAE.
- **You should submit your solutions to Gradescope by the deadline.** The first page of your solution should include your name (printed) followed by your UCLA ID (also printed) on the top of the page. Below, please include the following statement, followed by your signature and date (the first page should not contain anything else, i.e., no solutions on the first page):
  I assert, on my honor, that I have not received assistance of any kind from any other person, and have not used any non-permitted materials or technologies while working on the final. I agree with the rules summarized on the exam assignment cover page

  It is your responsibility to make sure that the files are uploaded correctly. In case of any technical difficulties you should notify your TA immediately. In particular, if you have issues with uploading your solutions to Gradescope, you may send your solutions to the TA to upload (before the deadline).
- **Collaborations on the final are not allowed.** You are under strict instructions not to discuss the exam or questions related to the exam with anybody. Please, be reminded of the Student Conduct Code (it can be found at www.deanofstudents.ucla.edu; see, in particular, Section 102.01 on academic dishonesty).
- **You are allowed to use the lecture notes posted online and the textbook**, as well as any other resources that were posted on the course webpages, while working on the exam. **You should not use any other resources, including online ones.**
- **You may use a calculator/computer for arithmetic only**, i.e., simplifying expression which involve $+$, $\times$,$-$, $/$. **You should not use computing systems (e.g., Mathematica or Matlab) while working on the problems.**
- Show your work on each problem. **Mysterious or unsupported answers will not receive credit**. A correct answer, unsupported by calculations and/or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **Organize your work**, in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering will receive very little credit.

GOOD LUCK!

1. (12 points) Compute the general solution of the following equation by:
   (a) Converting to an appropriate-sized linear system.
   (b) Solving the linear system.
   (c) Converting your answer back.

$$y''' - 2y'' - y' + 2y = 0.$$

2. (10 points) Solve the initial value problem of the form (i) $\mathbf{x}' = A\mathbf{x}$ (ii) $\mathbf{x}(0) = \mathbf{x}_0$, where $A = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

3. (12 points) Suppose $a, b, c \in \mathbb{R}$ are fixed constants such that $a < b < c$. Consider the following initial value problem:

$$y' + \frac{y}{t-c} \;=\; \frac{b-c}{(t-c)(t-a)}, \qquad y(b) \;=\; 1$$

   (a) Determine the interval of existence of the unique solution $y(t)$ (Note: this does not require any computation but may depend on the values of a, b, c).
   (b) Compute a valid integrating factor $\mu(t)$.
   (c) Compute the unique solution to the initial value problem.

4. (12 points) This problem covers the "double real root interesting case". We will find the general solution to the equation:

$$\mathbf{x}' \;=\; A\mathbf{x} \quad \text{where} \quad A \;=\; \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$$

   (a) Find the characteristic polynomial $p(\lambda)$ of A, then use this polynomial to determine the eigenvalues of A.
   (b) In this case, $\lambda := \lambda_1 = \lambda_2$ is an eigenvalue of multiplicity two. Since this is the interesting case, there is only one linearly independent eigenvector associated to λ. Find it.
   (c) Let $\mathbf{v}_1$ be the eigenvector found in step (b). Find a second vector $\mathbf{v}_2$ which is a solution to the matrix equation:
   $$(A - \lambda I)\mathbf{v}_2 \;=\; \mathbf{v}_1.$$
   (d) Write down the general solution:
   $$\mathbf{x}(t) \;=\; C_1 e^{\lambda t}\mathbf{v}_1 + C_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$$
   (e) Solve the initial value problem:
   $\mathbf{x}' = A\mathbf{x}$
   $\mathbf{x}(0) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

5. (15 points) Consider the following differential equation:

$$\frac{dy}{dt} = \frac{-2t\cos y - 3t^2 y}{t^3 - t^2 \sin y - y} \tag{$\dagger$}$$

   (a) Convert (†) into a differential form equation. Determine, with justification, whether your differential form equation is exact.

   (b) Determine the general solution of (†). You may leave your answer in implicit form.

   (c) Solve the following initial value problem:

$$\frac{dy}{dt} = \frac{-2t\cos y - 3t^2 y}{t^3 - t^2 \sin y - y}, \qquad y(4) = 0.$$

   You may leave your answer in implicit form and you do not need to specify the interval of existence.

6. (15 points) Find the general solution to the following homogeneous second-order linear differential equation:

$$y'' + 4y' + 4y = 0.$$

Use **variation of parameters** to find the solution to the following initial value problem (on the interval $(0, +\infty)$)(**no credit for using other methods**):

$$y'' + 4y' + 4y = t^{-2}e^{-2t}.$$

$$y(1) = 0, y'(1) = 0$$

7. (12 points) Consider the following initial value problem:

$$y' = \sin(y+t) + e^t - \sin(e^t + t), \qquad y(0) = 2$$

   (a) Show that there exists an interval $I \subseteq \mathbb{R}$ such that $0 \in I$, and a differentiable function $y : I \to \mathbb{R}$ which is a solution to this initial value problem.

   (b) Is the solution in (a) unique? Explain why or why not.

   (c) One of the following three things is true:

      1. $y(t) > e^t$ for all $t \in I$
      2. $y(t) < e^t$ for all $t \in I$
      3. Neither 1. nor 2.

   State which of the above is true and justify your answer.

8. (12 points) Use the method of **undetermined coefficients** to find particular solutions to each of the following (**no credit for using other methods**):

$$(a) \quad y'' - 2y' - 3y = e^{-t};$$

$$(b) \qquad y'' - 2y' - 3y = t.$$

Find the general solution to the following:

$$y'' - 2y' - 3y = 2e^{-t} - 3t.$$