Math 33B, Lec 2
Spring 2016
Exam 2
5/9/16
Time Limit: 50 Minutes

Name (Print):

Name (Sign):

Discussion Section:

This exam contains 6 pages, including this cover page and 5 problems.

You may *not* use books, notes, or any calculator on this exam.

Partial credit will only be awarded to answers for which an explanation and/or work is shown.

Problem	Points	Score
1	10	10
2	10	10
3	10	10
4	10	10
5	10	10
Total:	50	50

1. (10 points) (a) Find the general solution to the differential equation

$$y'' - 8y' + 16y = 0.$$

characteristic polynomial

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda - 4)(\lambda - 4) = 0$

$\lambda = 4$

$$\boxed{y(t) = C_1 e^{4t} + C_2 t e^{4t}}$$

check

$y' = 4C_1 e^{4t} + C_2 e^{4t} + 4C_2 t e^{4t}$

$y'' = 16C_1 e^{4t} + 4C_2 e^{4t} + 4C_2 e^{4t} + 16C_2 t e^{4t}$

$= 16C_1 e^{4t} + 8C_2 e^{4t} + 16C_2 t e^{4t}$

$16C_1 e^{4t} + 8C_2 e^{4t} + 16C_2 t e^{4t} - 32C_1 e^{4t} - 8C_2 e^{4t}$

$+ 32C_2 t e^{4t} + 16C_1 e^{4t} + 16C_2 t e^{4t}$

$= 0$ ✓

5

(b) Find the general solution to the differential equation

$$y'' - 8y' + 20y = 0.$$

characteristic polynomial

$\lambda^2 - 8\lambda + 20 = 0$

$\lambda = \frac{8 \pm \sqrt{64 - 80}}{2} = \frac{8 \pm \sqrt{-16}}{2}$

$= \frac{8 \pm 4i}{2} = 4 \pm 2i$

$$\boxed{y(t) = C_1 e^{4t} \cos 2t + C_2 e^{4t} \sin 2t}$$

5

2. Let $y_1(t) = t$ and $y_2(t) = t^3$.

(a) (3 points) Calculate the Wronskian of y_1 and y_2.

$$W(t) = \det \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} = t(3t^2) - t^3 = 3t^3 - t^3 = \boxed{2t^3}$$

(b) (7 points) Show that y_1 and y_2 are a fundamental set of solutions to the equation

$$t^2y'' - 3ty' + 3y = 0.$$

$y_1 = t \quad y_1' = 1 \quad y_1'' = 0$

$t^2(0) - 3t(1) + 3(t) = -3t + 3t = 0$ ✓

$y_2 = t^3 \quad y_2' = 3t^2 \quad y_2'' = 6t$

$t^2(6t) - 3t(3t^2) + 3(t^3) = 6t^3 - 9t^3 + 3t^3 = 0$ ✓

Both y_1 and y_2 are solutions to the differential equation, and the Wronskian is nonzero, so y_1 and y_2 form a fundamental set of solutions.

from part (a)

3. **(10 points)** Find the solution to the initial-value problem

$$y'' + y' - 12y = 0, \; y(0) = 1, \; y'(0) = 10.$$

characteristic polynomial

$\lambda^2 + \lambda - 12 = 0$

$(\lambda+4)(\lambda-3) = 0$

$\lambda = -4, 3$

General solution:

$y(t) = C_1 e^{-4t} + C_2 e^{3t}$

$y'(t) = -4C_1 e^{-4t} + 3C_2 e^{3t}$

$y(0) = 1 = C_1(1) + C_2(1) = C_1 + C_2 \Rightarrow 4C_1 + 4C_2 = 4$

$y'(0) = 10 = -4C_1(1) + 3C_2(1) = -4C_1 + 3C_2$

$-4C_1 + 3C_2 = 10$

$(+) \; 4C_1 + 4C_2 = 4$

$7C_2 = 14$

$C_2 = 2$

$C_1 + C_2 = C_1 + 2 = 1$

$C_1 = -1$

Particular solution:

$$\boxed{y(t) = -e^{-4t} + 2e^{3t}}$$

✓ 10

4. (10 points) Find the solution to the initial-value problem

$$y'' + y' - 12y = 6e^{2t} - 144t, \ y(0) = 1, \ y'(0) = 7.$$

From the previous problem:

$\lambda^2 + \lambda - 12 = (\lambda+4)(\lambda-3) = 0$

$\lambda = -4, 3$

$y_h = C_1 e^{-4t} + C_2 e^{3t}$

First forcing term:

$y_p = ae^{2t} \quad y_p' = 2ae^{2t} \quad y_p'' = 4ae^{2t}$

$4ae^{2t} + 2ae^{2t} - 12ae^{2t} = -6ae^{2t}$

we want $-6ae^{2t} = 6e^{2t}$, so $a = -1 \Rightarrow y_p = -e^{2t}$

Second forcing term:

$y_2 = at + b \quad y_2' = a \quad y_2'' = 0$

$0 + a - 12(at+b) = a - 12at - 12b = -12at + (a - 12b)$

we want $-(12a)t + (a-12b) = -144t$, so

$-12a = -144 \Rightarrow a = 12$

$a - 12b = 0 \Rightarrow 12 - 12b = 0 \Rightarrow b = 1$

$y_2 = 12t + 1$

$y(t) = C_1 e^{-4t} + C_2 e^{3t} - e^{2t} + 12t + 1$

$y'(t) = -4C_1 e^{-4t} + 3C_2 e^{3t} - 2e^{2t} + 12$

$y(0) = 1 = C_1(1) + C_2(1) - 1 + 1 = C_1 + C_2 = 1$

$y'(0) = 7 : -4C_1(1) + 3C_2(1) - 2 + 12 = -4C_1 + 3C_2 + 10 = 7$

$4C_1 + 4C_2 = 4$

$(+) \ -4C_1 + 3C_2 = -3$

$7C_2 = 1$

$C_2 = \frac{1}{7}$

$C_1 + C_2 = C_1 + \frac{1}{7} = 1 \Rightarrow C_1 = \frac{6}{7}$

$$\boxed{y(t) = \frac{6}{7}e^{-4t} + \frac{1}{7}e^{3t} - e^{2t} + 12t + 1}$$

5. A mass of 5 kg is attached to a large spring with a spring constant of $k = 20$ kg/s^2.

(a) **(7 points)** The system is then stretched 4 m from the spring-mass equilibrium and set to oscillating with an initial velocity of 6 m/s. Assume that it oscillates without damping. Write the differential equation describing the motion of the system, and use the solution of the equation to find the frequency, amplitude, and phase of the vibration. (You may leave your phase in terms of arctan.)

$m = 5kg$ $k = 20 kg/s^2$ $\mu = 0$ $y(0) = 4m$ $y'(0) = 6 m/s$

$5y'' + 20y = 0$

$y'' + 4y = 0$

characteristic polynomial

$\lambda^2 + 4 = 0$

$\lambda = \frac{\pm\sqrt{-16}}{2} = \frac{\pm 4i}{2} = \pm 2i$

$y(t) = a\cos(2t) + b\sin(2t)$

$y'(t) = -2a\sin(2t) + 2b\cos(2t)$

$y(0) = 4 = a(1) + b(0) = a = 4$

$y'(0) = 6 = -2a(0) + 2b(1) = 2b = 6 \Rightarrow b = 3$

$y(t) = 4\cos(2t) + 3\sin(2t) \Rightarrow$ $\boxed{\begin{array}{l}\omega = 2 \text{ rad/s} \\ A = 5m \\ \phi = \arctan\left(\frac{3}{4}\right)\end{array}}$

$A = \sqrt{4^2+3^2} = \sqrt{25} = 5$

$\phi = \arctan\left(\frac{b}{a}\right)$, if $a > 0$

(b) **(3 points)** Now suppose the system is placed in a viscous medium that supplies a damping constant that gives the system critical damping. Find the value of the damping constant μ for which the system is critically damped.

general harmonic motion equation is

$my'' + \mu y' + ky = 0$

characteristic polynomial

$m\lambda^2 + \mu\lambda + k = 0$

$\lambda = \frac{-\mu \pm \sqrt{\mu^2 - 4mk}}{2m}$, so we want $\mu^2 - 4mk = 0$

$\mu^2 = 4mk$

$\mu = \sqrt{4mk} = 2\sqrt{mk} = 2\sqrt{5kg \cdot \frac{20kg}{s^2}}$

$= 2\sqrt{100 \frac{kg^2}{s^2}}$

$= 2(10)\, kg/s = 20 kg/s$

$\boxed{\mu = 20 \frac{kg}{s}}$