Midterm 1

Name: SOLUTION

Student ID: ______________________

Signature: ______________________

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This is a closed-book exam. Do not use notes, books, papers, or electronic devices of any kind. Do all work on the sheets provided. Do not use your own paper or blue books. If you need more space for your solution, use the back of each page; you may request extra paper. Be sure to state clearly if you are continuing on a different page and label the problems well.

Do all 5 problems. For full credit, you must show all your work. Do not worry about oversimplifying your answers. Please clearly indicate your final answer, for example by putting a box around it.

Problem	Out of	Points
1		
2		
3		
4		
5		
Σ		

(1) (points) Radioactive substances decay according to the differential equation

$$N' = -\lambda N$$

where $N(t)$ is the amount of the substance remaining at time t and λ is a constant, called the decay constant.

(a) Solve the above differential equation to find N(t).

$$\frac{dN}{dt} = -\lambda N \Rightarrow \int \frac{dN}{N} = \int -\lambda \, dt \Rightarrow \ln|N| = \cancel{e^{-\lambda t}} -\lambda t + C$$

$$\Rightarrow \boxed{N = Ae^{-\lambda t}}$$

(b) Show that after a period of $T_\lambda = \frac{1}{\lambda}$, the material has decreased to e^{-1} of its original value. T_λ is called the **time constant**.

$$N(T_\lambda) = Ae^{-\lambda \cdot 1/\lambda} = Ae^{-1}$$

$N(0) = A$ since $Ae^{-1} = A(e^{-1})$ this is true.

(c) A certain radioactive substance has half-life of 10 hours. Compute its time constant.

$$\underbrace{N(10)}_{= A/2} = Ae^{-10\lambda} \Rightarrow \frac{1}{2} = e^{-10\lambda}$$

$$\Rightarrow -\ln(2) = -10\lambda$$

$$\Rightarrow \lambda = \frac{\ln(2)}{10}$$

so $\boxed{T_\lambda = 1/\lambda = 10/\ln(2)}$

(2) (points) Only one of the following differential equations is linear. Determine which one it is and solve it using any method from class.

(a) $y' = y + y^2$

(b) $y' + \dfrac{2y}{t} = \dfrac{\cos t}{t^2}$

(c) $y' - 2ty = \cos y$

(d) $y' = t/y$

(b) is the linear one. $a(t) = \frac{-2}{t}$, $f(t) = \frac{\cos t}{t^2}$

Integrating factor: $e^{-\int a(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = t^2$

$t^2(y' + 2y/t) = t^2 \cdot \frac{\cos t}{t^2} = \cos t$

$\shortparallel$

$(t^2y)' = \cos t \Rightarrow t^2 y = \int \cos t + C$

$= \sin t + C$

So $\boxed{y(t) = t^{-2}\sin t + Ct^{-2}}$ is the general solution.

(3) (points) A tank initially contains 100 gallons of pure water. Water begins entering the tank through two pipes: through pipe A at 6 gal/min and through pipe B at 4 gal/min. Simultaneously, a drain is opened at the bottom of the tank, letting solution flow out at 10 gal/min.

(a) Supervisors quickly discover that the water coming through pipe B is contaminated, containing 0.5 lb of pollutant per gallon. If this process runs for 10 minutes, how many pounds of pollutant are in the tank after this 10-minute period? (Call this number x_p for use in part (b) of this problem)

$x(t)$ = # of lbs of pollutant at time t.

$$x' = (\text{rate in}) - (\text{rate out}) = 2 - 10\frac{x}{100}$$

$$x' = \frac{20-x}{10} \Rightarrow \int \frac{dx}{20-x} = \int \frac{dt}{10}$$

$$\Rightarrow -\ln|20-x| = \frac{t}{10} + C \quad 20 - x = Ae^{-t/10} \quad x(t) = 20 - Ae^{-t/10}$$

$$x(0) = 0 \Rightarrow A = 20$$

$$\Rightarrow x(t) = 20 - 20e^{-t/10}$$

We want $x(10) = \boxed{20 - 20e^{-1} = x_p}$

(b) The supervisors shut off pipe B after this 10-minute period, allowing pipe A and the drain to function as before. How much pollutant remains after 10 minutes of this new process? (Use the notation x_p instead of the potentially complicated number from part (a)).

$x(t) = \#$ lbs pollutant at time t.

$$x' = (\text{rate in}) - (\text{rate out})$$

$$= 0 - 10 \cdot \frac{x(t)}{V(t)} \qquad V(t) = 100 - 4t$$

$$x' = -10 \frac{x}{100-4t} \Rightarrow \frac{dx}{dt} = -10 \frac{x}{100-4t}$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{-10\,dt}{100-4t} \qquad (u = 100-4t, du = -4\,dt)$$

$$\Rightarrow \ln|x| = \frac{-10}{-4} \ln|100-4t| + C$$

$$\Rightarrow x = A(100-4t)^{10/4} \qquad x_0 = x_p \text{ from before}$$

$$\underset{\underset{x_p}{\shortparallel}}{x(0)} = A(100)^{10/4} \Rightarrow A = x_p (100)^{-10/4}$$

So $x(t) = x_p \cdot 100^{-10/4} (100-4t)^{10/4}$

We want $\boxed{x(10) = x_p \cdot 100^{-10/4} (60)^{10/4}}$

(4) (points) Solve the following differential equation. Hint: Check for exactness, homogeneity. If neither holds, assume that the integrating factor is a function of either x alone or y alone.

$$(x+y)\sin y\ dx + (x\sin y + \cos y)\ dy = 0$$

$$= P dx + Q dy$$

$\frac{\partial P}{\partial y} = (x+y)\cos y + \sin y$, $\frac{\partial Q}{\partial x} = \sin y$ so not exact.

Also not homogeneous.

Since $g = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{(x+y)\cos y}{(x+y)\sin y} = \frac{\cos y}{\sin y}$

is a function of y alone, the integrating factor is

$\mu(y) = e^{-\int g(y)dy}$ $\quad \int \frac{\cos y}{\sin y} dy$ $\quad u = \sin y$, $du = \cos y\, dy$

$\boxed{\Rightarrow \mu(y) = \frac{1}{\sin y}.}$ $\quad = \int \frac{du}{u} = \ln|u| = \ln|\sin y|$

This turns our ODE into $(x+y)dx + \left(x + \frac{\cos y}{\sin y}\right)dy = 0$, which is easily checked to be exact.

$F(x,y) = \int P dx + \varphi(y) \Rightarrow F(x,y) = \frac{1}{2}x^2 + yx + \varphi(y)$.

$\varphi'(y) = \frac{\partial}{\partial y}\int P dx - Q = x - \left(x + \frac{\cos y}{\sin y}\right) = \frac{-\cos y}{\sin y}$

So $\varphi = \int \frac{-\cos y}{\sin y} dy = -\ln|\sin y|$ (as above).

and the general solution is

$$\boxed{F(x,y) = \frac{1}{2}x^2 + yx - \ln|\sin y| = C}$$

(5) (points) Consider the initial value problem

$$x' = -x^2 \cos t, \quad x(\frac{\pi}{2}) = 1$$

(a) Use the existence and uniqueness theorems to prove that a solution to this differential equation exists and is unique.

$x' = f(x,t) = -x^2 \cos t$. $f(x,t)$ is defined and continuous everywhere so a solution exists.

$\frac{\partial f}{\partial x} = -2x\cos t$ is also defined and continuous everywhere, hence the solution is unique.

(b) Solve this differential equation and give the interval of existence of the solution.

$$\frac{dx}{dt} = -x^2 \cos t \Rightarrow \int \frac{-dx}{x^2} = \int \cos t\, dt$$

$$\Rightarrow x^{-1} = \sin t + C \Rightarrow x = \frac{1}{\sin t + C}$$

$x(\pi/2) = 1 \Rightarrow C = 0$ $\quad \boxed{x(t) = \frac{1}{\sin t}.}$

$\boxed{\text{Interval of existence: } (0, \pi)}$