Midterm 2

Name: KEY

Student ID: ____________________

Signature: ____________________

Check your section: __ 1a (Tu) __ 1b (Th) TA: Neel Tiruviluamala

__ 1c (Tu) __ 1d (Th) TA: Eric Radke

This is a closed-book exam. Do not use notes, books, papers, or electronic devices of any kind. Do all work on the sheets provided. Do not use your own paper or blue books. If you need more space for your solution, use the back of each page; you may request extra paper. Be sure to state clearly if you are continuing on a different page and label the problems well.

Do all 5 problems. For full credit, you must show all your work. Do not worry about oversimplifying your answers. Please clearly indicate your final answer, for example by putting a box around it.

Problem	Out of	Points
1	9	
2	9	
3	12	
4	8	
5	12	
Σ	50	

(1) (9 points) For each of parts (a)-(c), determine whether or not the statement is true or false. If true, show why. If false, explain why not.

(a) (3 points) There exists a differential equation of the form $y'' + p(t)y' + q(t)y = 0$ such that $y_1(t) = e^t$ and $y_2(t) = t^2$ are two solutions on the interval $(-10, 10)$.

False. If one did exist, their Wronskian would be either identically equal to 0 on $(-10,10)$ or never equal to 0 there. But $W = y_1 y_2' - y_1' y_2 = e^t \cdot 2t - t^2 e^t = te^t(2-t)$ equals 0 at $t = 0, 2$ and nowhere else.

(b) (3 points) If A and $\mathbf{v}$ are defined as

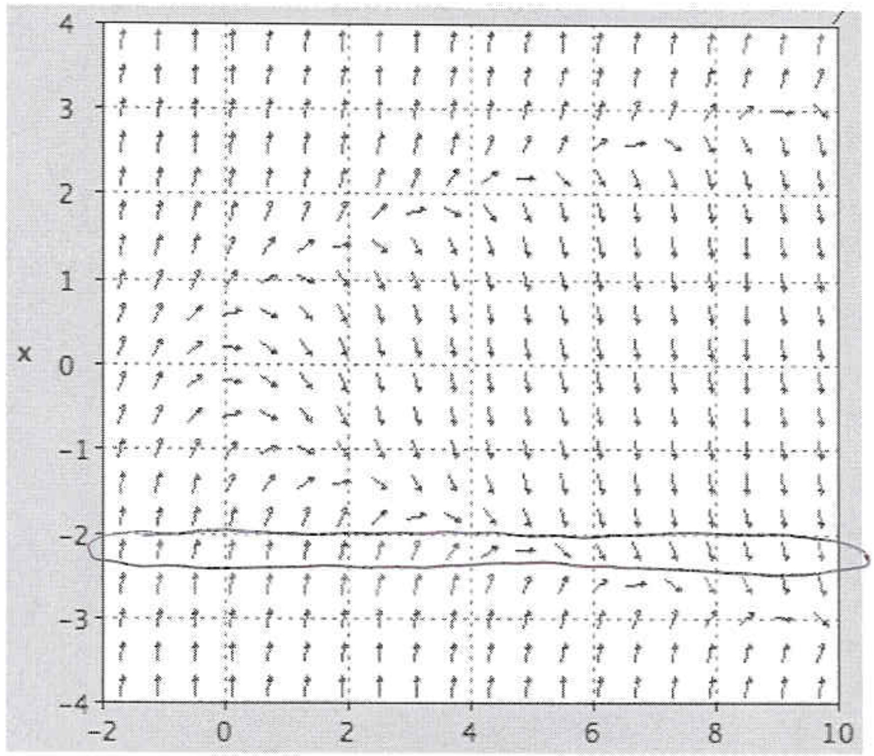
$$A = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

then $\mathbf{v}$ is an eigenvector of A with associated eigenvalue 2.

True. To check, it is enough to note that

$$Av = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 2 \cdot v.$$

(c) (3 points) The differential equation giving the following direction field is autonomous.

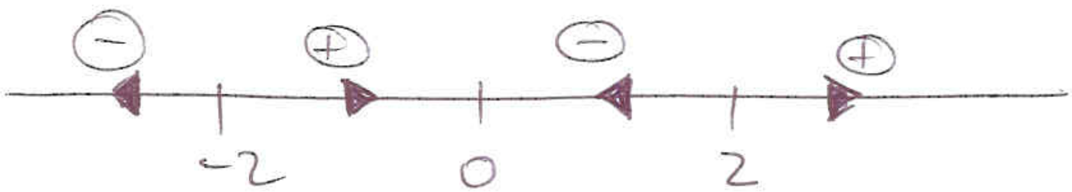


False. If it were autonomous, the arrows would not depend on t; i.e. they would be all in the same direction in the circled region above.

(2) (9 points) In this problem we consider the differential equation $x' = x^3 - 4x$.

(a) (6 points) Draw a phase line for this autonomous differential equation. Classify each equilibrium point as either unstable or asymptotically stable.

$f(x) = x^3 - 4x = x(x^2-4) = x(x-2)(x+2)$

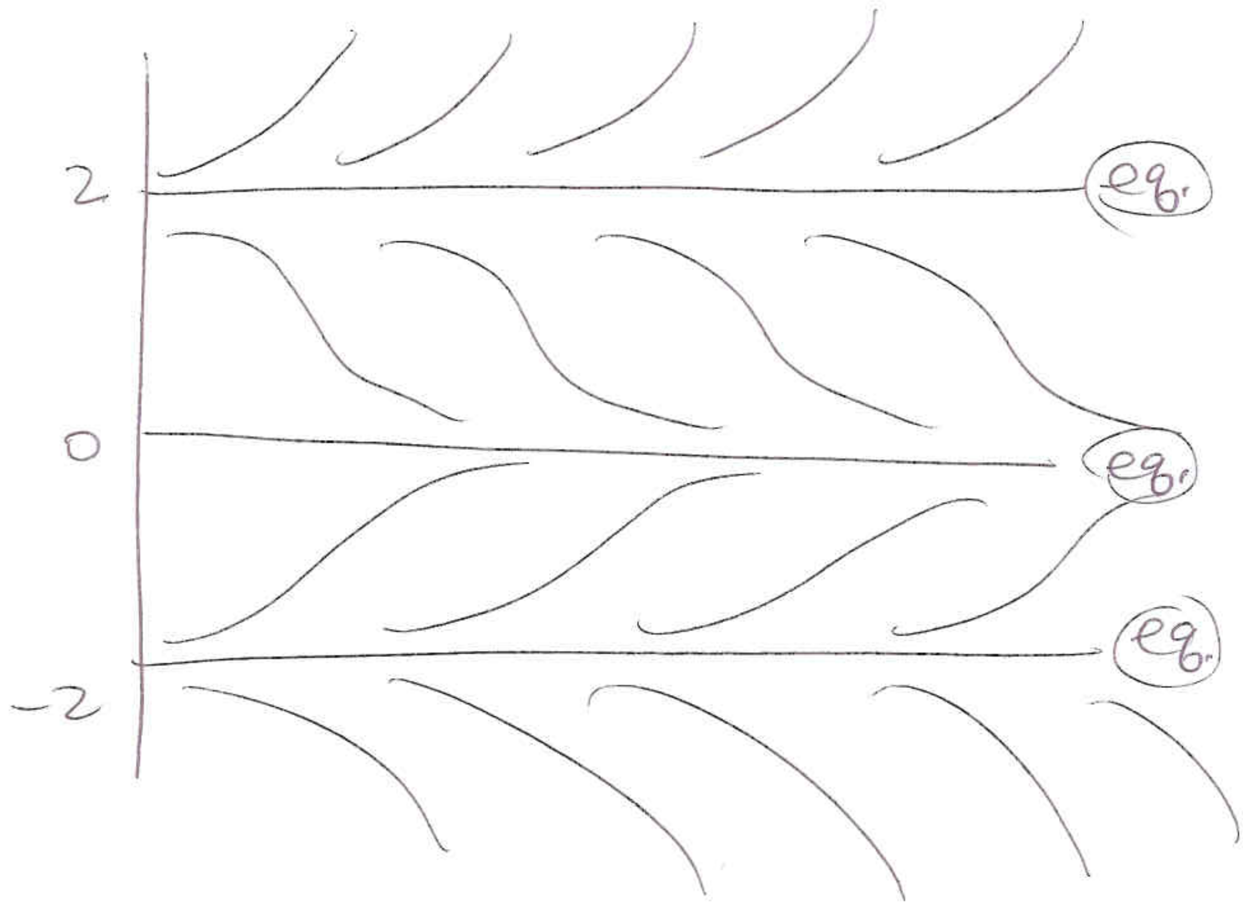


The three equilibria are 0, 2, −2.

0 is stable while

−2, 2 are unstable.

(b) (3 points) Sketch some solution curves in the tx-plane, indicating the equilibrium solutions.



(3) (12 points) A 2 kg mass, when attached to a certain spring, stretches the spring 20 cm. Assume this system is unforced.

(a) (3 points) Calculate the spring constant for this spring.

$F = mg = kx$ so $2 \cdot 9.8 = k \cdot 0.2$

$\boxed{\Rightarrow k = 98}$

(b) (3 points) Find the damping constant μ for which there is critical damping.

$2y'' + \mu y' + 98y = 0$

$y'' + \mu/2\, y' + 49y = 0$ critically damped when $c = \omega_0$.

$y'' + 2cy' + \omega_0^2 y = 0$

$c = \mu/4 = \sqrt{49} = 7 = \omega_0$

$\boxed{\text{so } \mu = 28}$

(c) (6 points) Find the solution to this system given the initial conditions $y(0) = 1$ m, $y'(0) = 2$ m/s.

$$y'' + 14y' + 49y = 0$$

$$\lambda^2 + 14\lambda + 49 = 0 \Rightarrow (\lambda+7)^2 = 0 \Rightarrow \lambda = -7$$

so $y(t) = C_1 e^{-7t} + C_2 t e^{-7t}$

$$\begin{aligned} y(0) &= C_1 = 1 \\ y'(0) &= -7C_1 + C_2 = 2 \end{aligned} \Rightarrow \begin{aligned} C_1 &= 1 \\ C_2 &= 2+7 = 9 \end{aligned}$$

so our solution is

$$\boxed{y(t) = e^{-7t} + 9te^{-7t}}$$

(4) (8 points) Find a particular solution to the differential equation

$$y'' + 9y = \sec(3t)$$

(Hint: Recall that $\int \tan(t)dt = \ln|\sec(t)| + C$)

We use variation of parameters.

$$y'' + 9y = 0 \Rightarrow y_1 = \sin(3t),\ y_2 = \cos(3t)$$

$$y_p = v_1 y_1 + v_2 y_2 = v_1 \sin(3t) + v_2 \cos(3t)$$

$$y_p' = 3v_1 \cos(3t) - 3v_2 \sin(3t) + v_1' \sin(3t) + v_2' \cos(3t)$$

Set $v_1' \sin(3t) + v_2' \cos(3t) = 0$

Then $y_p'' = -9v_1 \sin(3t) - 9v_2 \cos(3t) + 3v_1' \cos(3t) - 3v_2' \sin(3t)$

and $y_p'' + 9y_p = 3v_1' \cos(3t) - 3v_2' \sin(3t)$.

Two equations to solve:

$$3v_1' \cos(3t) - 3v_2' \sin(3t) = \sec(3t)$$

$$v_1' \sin(3t) + v_2' \cos(3t) = 0$$

solve to get $v_1' = \frac{1}{3}$, $v_2' = \frac{-1}{3}\tan(3t)$

$$\Rightarrow v_1 = \tfrac{1}{3}t,\ v_2 = \frac{-1}{9}\ln|\sec(3t)|$$

so our particular solution is

$$\boxed{y_p(t) = \frac{1}{3}t\sin(3t) - \frac{1}{9}\ln|\sec(3t)|\cos(3t)}$$

(5) (12 points)

(a) (4 points) Prove that the imaginary part of the solution of $z'' + z' + z = te^{it}$ is a solution of $y'' + y' + y = t\sin(t)$. (Hint: let $z(t) = x(t) + iy(t)$)

$z(t) = x(t) + iy(t)$

$z'(t) = x'(t) + iy'(t)$

$z''(t) = x''(t) + iy''(t)$

and $te^{it} = t(\cos t + i\sin t) = t\cos t + it\sin t$.

$z'' + z' + z = x'' + x' + x + i(y'' + y' + y)$

So equating imaginary parts, we get

$\boxed{y'' + y' + y = t\sin t}$ ✓

(b) (8 points) Use this idea to find a particular solution of $y'' + y' + y = t\sin(t)$. (Hint: solve $z'' + z' + z = te^{it}$ using the method of undetermined coefficients.)

$f(t) = te^{it} \Rightarrow f'(t) = tie^{it} + e^{it} = (it+1)e^{it}$

So we look for a solution $z(t)$ of the form $(At+B)e^{it}$.

$z' = (At+B)ie^{it} + Ae^{it} = e^{it}(Ait + Bi + A)$

$z'' = e^{it}(Ai) + ie^{it}(Ait + Bi + A) = e^{it}(-At + 2Ai - B)$

$\Rightarrow z'' + z' + z = e^{it}(-At + 2Ai - B + Ait + Bi + A + At + B)$

$= e^{it}(Ait + 2Ai + Bi + A)$

this is supposed to equal te^{it}.

Hence, equating coefficients,

$Ai = 1, \quad 2Ai + Bi + A = 0.$

Solve to get $A = -i$, $B = 1+2i$.

So $z(t) = (-it + (1-2i))e^{it}$. What is the imaginary part?

$= (i(-t+2) + 1)(\cos t + i\sin t)$

$= ((t-2)\sin t + \cos t) + i((-t+2)\cos t + \sin t)$

So $\boxed{y_p(t) = (-t+2)\cos t + \sin t}$ is a particular solution to $y'' + y' + y = t\sin t$