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Final Exam

Name: ______________________

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__ 1c (Tu) __ 1d (Th) TA: Eric Radke

This is a closed-book exam. Do not use notes, books, papers, or electronic devices of any kind. Do all work on the sheets provided. Do not use your own paper or blue books. If you need more space for your solution, use the back of each page; you may request extra paper. Be sure to state clearly if you are continuing on a different page and label the problems well.

Do all 10 problems. For full credit, you must show all your work. Do not worry about oversimplifying your answers. Please clearly indicate your final answer, for example by putting a box around it.

Problem	Out of	Points	Problem	Out of	Points
1	12		6	10	
2	8		7	9	
3	20		8	8	
4	10		9	18	
5	9		10	10	

(1) (12 points) For each of parts (a)-(d), determine whether the statement is true or false. In either case, thoroughly explain your answer.

(a) (3 points) The differential equation modeling Newton's law of cooling,

$$\frac{dT}{dt} = -k(T - A)$$

where k and A are constants, is a separable differentiable equation.

It is $\boxed{\text{true}}$: we can separate the variables to make it

$$\cancel{dt}\ \frac{dT}{T-A} = -k\,dt.$$

(In fact, this equation is autonomous and all autonomous equations are separable).

(b) (3 points) Two differentiable functions $f(t)$ and $g(t)$ on an interval (a, b) are linearly independent if and only if their Wronskian is nonzero on that interval.

<u>FALSE</u>. ~~Example $f(t)=t,\ g(t)=\sin(t)$~~.

~~$f'g - g'f = 1\cdot\sin(t) - t\cos(t)$~~

Better example: $f(t)=t^2$, $g(t)=e^t$. Certainly linearly independent but $f'g-g'f = 2te^t - t^2e^t = t(2-t)e^t = 0$ at $t=0$ and $t=2$.

This fact is ~~only~~ true if f and g are solutions to a certain differential equation, but is not true in general.

(c) (3 points) If one tries to solve the differential equation $x' = (\sin^2 t)x + t^2$ using the integrating factor method, the integrating factor would be

$$u(t) = e^{\int \sin^2 t \, dt}$$

False. The integrating factor is $e^{-\int a(t)dt}$ which in this case would be $e^{-\int \sin^2 t \, dt}$.

(d) (3 points) Every autonomous first-order differential equation has an equilibrium solution.

False. $x' = x^2 + 1$ has no equilibrium solution.

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(2) (points) Find the solution to the following initial value problem. What is the interval of existence of the solution?

$$ty' + y = t^2 \quad y(1) = 2$$

$y' + \frac{1}{t}y = t \Rightarrow y' = \frac{-1}{t}y + t$ Using VOP:

$y_h = e^{-\int \frac{1}{t}dt} = e^{-\ln|t|} = 1/t.$

$v' = f/y_h = t/(1/t) = t^2 \Rightarrow v = \frac{1}{3}t^3 + C$

$\Rightarrow y = (\frac{1}{3}t^3 + C) \cdot 1/t = \boxed{\frac{1}{3}t^2 + C/t.}$ Using $y(1) = 2$, we get

$2 = y(1) = \frac{1}{3} + C \Rightarrow C = 5/3$

so $\boxed{y(t) = \frac{1}{3}t^2 + \frac{5}{3t}}$.

The interval of existence cannot contain $t=0$ and must contain $t=1$, so it is $\boxed{(0,\infty)}$.

(3) (20 points) The picture on the board shows two tanks, each containing ~~50 gallons of~~ a salt solution. Tank A contains 50 gallons of solution in which is dissolved 20 lb of salt. Tank B contains 25 gallons of solution in which is contained 30 lb of salt. Salt water (concentrated at 2 lb/gal) pours into tank A at 5 gal/min. ~~There are two pipes connecting tank A to tank B. The first pumps salt solution from tank B into tank A at 2 gal/min~~. There is a drain at the bottom of tank A. Solution leaves tank A via this drain at a rate of 5 gal/min and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B, also at the rate of 5 gal/min.

(a) (4 points) Set up equations modeling this initial value problem. Why can we not set this up in matrix-vector form?

$x_1' = (\text{rate in}) - (\text{rate out}) = 10 - \frac{5x_1}{50} = 10 - x_1/10 \qquad x_1(0) = 20$

$x_2' = (\text{rate in}) - (\text{rate out}) = \frac{5x_1}{50} - \frac{5x_2}{25} = \frac{x_1}{10} - \frac{x_2}{5} \qquad x_2(0) = 30$

We cannot set this up in matrix-vector form because x_1' is not of the form $a_{11}x_1 + a_{12}x_2$.

(b) (6 points) If set up properly, one of these equations can be solved without considering the other. Solve this one, using your initial condition.

We can solve $x_1' = \frac{-x_1}{10} + 10$. Using VOP: $x_h = e^{-\int \frac{1}{10}dt} = e^{-t/10}$

$v' = f/x_h = \frac{10}{e^{-t/10}} = 10e^{t/10} \Rightarrow v = \int 10e^{t/10}dt = 100e^{t/10} + C$

so $x_1 = v\,x_h = (100e^{t/10} + C)e^{-t/10} = \boxed{100 + Ce^{-t/10}}$

$x_1(0) = 20$ so $C = -80$: $\boxed{x_1(t) = 100 - 80e^{-t/10}}$

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(c) (points) Substitute the result from part (b) into your other equation and solve that one, using the initial condition.

$x_2' = \frac{x_1}{10} - \frac{x_2}{5} = (10 - 8e^{-t/10}) - \frac{x_2}{5} = \frac{-1}{5}x_2 + (10 - 8e^{-t/10})$. Again with VOP:

$x_h = e^{-\int \frac{1}{5} dt} = e^{-t/5}$. $v' = \frac{f}{x_h} = (10 - 8e^{-t/10}) \cdot e^{t/5} = 10e^{t/5} - 8e^{t/10}$

so $v = 50e^{t/5} - 80e^{t/10} + C$. $x_2 = v \cdot x_h = (50e^{t/5} - 80e^{t/10} + C) \cdot e^{-t/5}$

$= \boxed{50 - 80e^{-t/10} + Ce^{-t/5}}$ using $x_2(0) = 30$, we get $C = 60$

so $\boxed{x_2(t) = 50 - 80e^{-t/10} + 60e^{-t/5}}$

4

(d) (points) What is the long-term behavior of the amount of salt in each tank, i.e. what happens to your solutions as $t \to \infty$? Explain why this makes sense.

$\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} (100 - 80e^{-t/10}) = 100$ ← tends to 100 lb of salt

$\lim_{t\to\infty} x_2(t) = \lim_{t\to\infty} (50 - 80e^{-t/10} + 60e^{-t/5}) = 50$ ← tends to 50 lb of salt.

This makes sense because the solution is being replaced with solution concentrated at 2 lb/gal. So the ~~100 gallon~~ 50 gallon tank tends to 100 lb of salt and the 25 gallon tank tends to 50 lb of salt.

10

(4) (points) Find the general solution to the following differential equation:

$$(x^2 - 2y^2)dx + x^2 dy = 0$$

Note that both $f(x,y) = x^2 - 2y^2$ and $g(x,y) = x^2$ are homogeneous of degree 2:

$f(tx, ty) = (tx)^2 - 2(ty)^2 = t^2(x^2 - 2y^2) = t^2 f(x,y)$

$g(tx, ty) = (tx)^2 = t^2 x^2 = t^2 g(x,y)$

So we make the substitution $y = xv$, $dy = v\,dx + x\,dv$.

Hint:

$$\frac{1}{(2t+1)(t-1)} = \frac{-1}{3}\frac{1}{t+\frac{1}{2}} + \frac{1}{3}\frac{1}{t-1}$$

We get $(x^2 - 2x^2v^2)dx + x^2(v\,dx + x\,dv) = 0$

$= (x^2 - 2x^2v^2 + x^2v)\,dx + x^3 dv$

$= (x^2(1 - 2v^2 + v))\,dx + x^3 dv$: divide through by $(1 - 2v^2 + v)\,x^3$:

$\Rightarrow \frac{1}{x}dx + \frac{1}{1 - 2v^2 + v}dv = 0 \Rightarrow \frac{1}{x}dx + \frac{-1}{2v^2 - v - 1}dv = 0$

$\Rightarrow \frac{1}{x}dx + \left(\frac{-1}{3}\frac{1}{v+\frac{1}{2}} + \frac{1}{3}\frac{1}{v-1}\right)dv = 0$

$\Rightarrow \ln|x| + \frac{-1}{3}\ln|v + \tfrac{1}{2}| + \frac{1}{3}\ln|v-1| + C = 0$. Substitute $v = y/x$:

$$\boxed{\ln|x| - \frac{1}{3}\ln\left|\frac{y}{x} + \frac{1}{2}\right| + \frac{1}{3}\ln\left|\frac{y}{x} - 1\right| + C = 0}$$

9

(5) (points)

4

(a) (points) Given a first-order differential equation $x' = f(t, x)$, what conditions on f must be satisfied for the differential equation to have a unique solution?

$f(t,x)$ and its partial derivative $\frac{\partial f}{\partial x}$ must both be continuous on some rectangle R in the tx-plane. Then if $x(t_0)=x_0$ there is only one solution.

5

(b) (points) Suppose that y is a solution to the initial value problem

$$y' = (y-2)\sin(e^{ty}) \quad y(0) = 1$$

Show that $y < 2$ for all t for which y is defined.

This initial value problem satisfies the above conditions so it has a unique solution.

Note that $y=2$ satisfies $y'=(y-2)\sin(e^{ty})$ so $y=2$ is a solution. Since solutions cannot cross (by uniqueness) and $y(0)=1<2$, $y(t)<2$ for all t.

10

(6) (points) A mass of 10 kg is attached to a large spring with spring constant k=40 kg/s^2. It is then stretched $\sqrt{3}$ meters from the spring-mass equilibrium and given an initial velocity of 2 m/s. Assuming it oscillates without damping, find the frequency, amplitude, and phase of the vibration. Sketch the solution.

$my'' + ky = 0$, $y(0) = \sqrt{3}$, $y'(0) = 2$

$\Rightarrow 10y'' + 40y = 0 \Rightarrow y'' + 4y = 0 \Rightarrow y(t) = A\sin(2t) + B\cos(2t).$

Using our initial conditions: $y(0) = 0 + B = \sqrt{3}$ so $B = \sqrt{3}$

$y'(0) = 2 = 2A \Rightarrow A = 1$

$\Rightarrow \boxed{y(t) = \sin(2t) + \sqrt{3}\cos(2t)}$

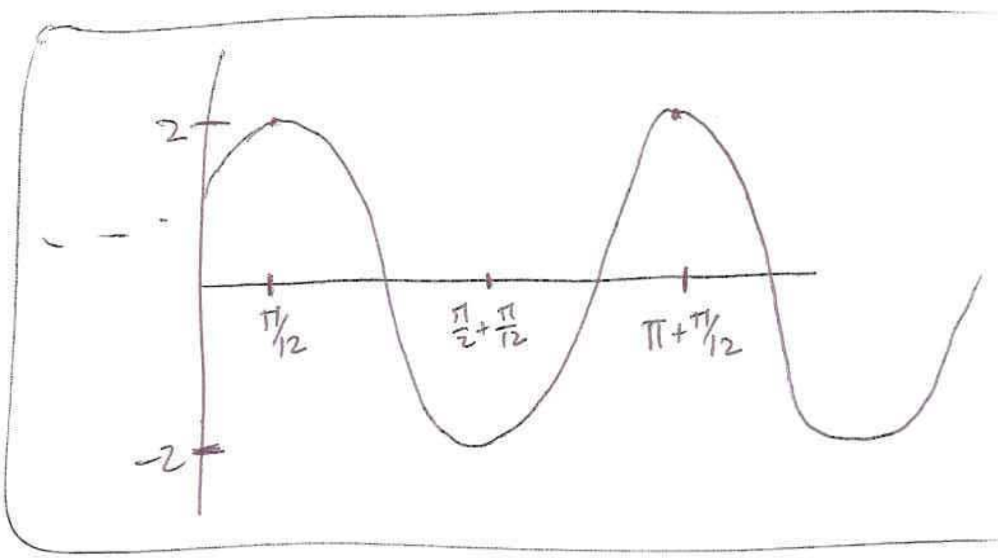
$= R\cos(\omega_0 t - \phi)$: $R = \sqrt{1+3} = 2$

$\tan\phi = \frac{1}{\sqrt{3}}$; $\phi = \pi/6$

$\omega_0 = 2$

$= 2\cos(2t - \pi/6) = 2\cos(2(t - \frac{\pi}{12}))$

Amplitude: 2
Phase: $\pi/6$
Frequency: 2



(7) (9 points)

(a) (3 points) Verify that $y_1(t) = t$ and $y_2(t) = t^{-1}$ are solutions to the homogeneous equation

$$t^2y''(t) + ty' - y = 0$$

$y_1' = 1,\ y_1'' = 0;\ y_2' = -t^{-2},\ y_2'' = 2t^{-3}$

$t^2y_2'' + ty_2' - y_2 = t^2(2t^{-3}) + t(-t^{-2}) - t^{-1} = 2t^{-1} - t^{-1} - t^{-1} = 0 \checkmark$

$t^2y_1'' + ty_1' - y_1 = t^2(0) + t - t = 0 \checkmark$

(b) (6 points) Use variation of parameters to find the general solution to

$$t^2y''(t) + ty' - y = t^3$$

$\Rightarrow y'' + \frac{1}{t}y' - \frac{1}{t^2}y = t$

$y_1(t)y_2'(t) - y_1'(t)y_2(t) = t(-t^{-2}) - t^{-1} = -2t^{-1}$

$V_1 = -\int \frac{1}{-2t^{-1}}\,dt = \frac{1}{2}\int t\,dt = t^2/4$

$V_2 = \int \frac{t^2}{-2t^{-1}}\,dt = \frac{-1}{2}\int t^3\,dt = -t^4/8$

$\Rightarrow y_p = V_1y_1 + V_2y_2 = \frac{t^3}{4} - t^3/8 = t^3/8$

and the general solution is $\boxed{y(t) = C_1t + C_2t^{-1} + t^3/8}$

(8) (8 points) Find the solution of the initial value problem $\mathbf{y}' = A\mathbf{y}$, where $A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ with initial condition $\mathbf{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvalues: $-3, -1$.

$(A+3I)v = \begin{pmatrix} 0 & -6 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow b = 0$ so $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector.

$(A+I)v = \begin{pmatrix} -2 & -6 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow \begin{matrix} -2a-6b=0 \\ a=-3b \end{matrix}$ so $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is an eigenvector

The general solution is thus

$$\vec{y}(t) = C_1 e^{-3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\vec{y}(0) = \begin{pmatrix} C_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3C_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} C_1 - 3C_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow C_2 = 1, C_1 = 4$$

So our solution is:

$$\boxed{\vec{y}(t) = 4e^{-3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}}$$

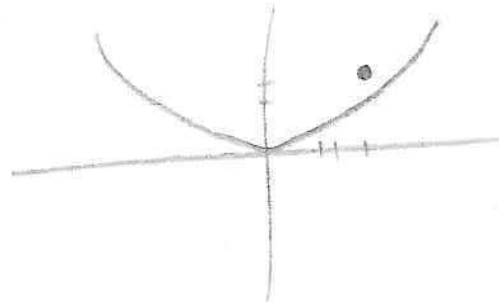
(9) (18 points) In this problem, let A be the matrix $\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$.

(a) (5 points) Determine where in the trace-determinant plane the system $\mathbf{y}' = A\mathbf{y}$ fits. What type of system is this?

$T = 1+3 > 0$, $D = 3+2 = 5 > 0$

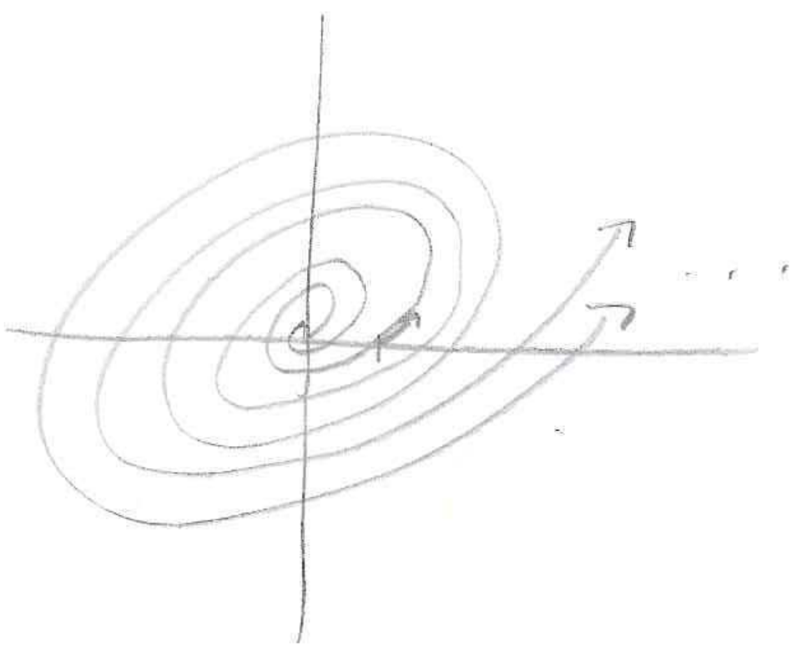
$T^2 - 4D = 16 - 20 = -4 < 0$

This is a spiral source.



(b) (4 points) Give a rough sketch of the phase portrait for this system, based on the above analysis.

Check $(1,0)$: $y' = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



(c) (9 points) Find a fundamental set of real solutions for this system.

$$ch_A(t) = \det(tI - A) = \det\begin{pmatrix} t-1 & 2 \\ -1 & t-3 \end{pmatrix} = (t-1)(t-3)+2$$
$$= t^2 - 4t + 5$$

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4\pm}{2}$$
$$= 2 \pm i$$

$$\lambda I - A = \begin{pmatrix} 1+i & 2 \\ -1 & -1+i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$ Example eigenvector: $\begin{pmatrix} -2 \\ 1+i \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ i \end{pmatrix}$
$$= \begin{pmatrix} -2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So our general solution is

$$\boxed{\vec{y}(t) = C_1 e^{2t}\left(\cos t\begin{pmatrix} -2 \\ 1 \end{pmatrix} - \sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + C_2 e^{2t}\left(\sin t\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}$$

Alternatively, we say that a fund. set of real solutions is given by

$$\boxed{\begin{aligned} \vec{y}_1(t) &= e^{2t}\left[\cos t\begin{pmatrix} -2 \\ 1 \end{pmatrix} - \sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \\ y_2(t) &= e^{2t}\left[\sin t\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \end{aligned}}$$

(10) (10 points) Solve the system of differential equations $\mathbf{y}' = A\mathbf{y}$ where A is the matrix

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$ch_A(t) = \det(tI - A) = \det\begin{pmatrix} t-1 & 1 & -3 \\ 0 & t-1 & 0 \\ 0 & -1 & t+1 \end{pmatrix}$$

$$= \left|\begin{matrix} t-1 & 1 & -3 \\ 0 & t-1 & 0 \\ 0 & -1 & t+1 \end{matrix}\right| \begin{matrix} t-1 & 1 \\ 0 & t-1 \\ 0 & -1 \end{matrix} \Rightarrow \boxed{(t-1)^2(t+1)} \qquad \text{Let } \lambda_1 = -1,\ \lambda_2 = +1$$

$\boxed{\lambda_1}$ $\lambda_1 I - A = -I - A = \begin{pmatrix} -2 & 1 & -3 \\ 0 & -2 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{matrix} -2a + b - 3c = 0 \\ -2b = 0 \\ -b = 0 \end{matrix}$

So an eigenvector here is given by $\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$, and a solution by $\boxed{e^{-t}\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}}$

$\boxed{\lambda_2:}$ $\lambda_2 I - A = I - A = \begin{pmatrix} 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{matrix} b - 3c = 0 \\ -b + 2c = 0 \\ \Rightarrow b = c = 0 \end{matrix}$

so eigenvectors look like $\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$: <u>not enough</u>.

$$(\lambda_2 I - A)^2 = \begin{pmatrix} 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 3 & -6 \\ 0 & 0 & 0 \\ 0 & -2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{matrix} 3b - 6c = 0 \\ -2b + 4c = 0 \end{matrix}$$

solutions here look like $\begin{pmatrix} a \\ 2c \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

$$e^{tA}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e^t\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t(A - I)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right] = e^t\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right] = \boxed{e^t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}.$$

$$e^{tA}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = e^t\left[\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + t(A - \lambda I)\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right] = e^t\left[\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}\right] = \boxed{e^t\begin{pmatrix} t \\ 2 \\ 1 \end{pmatrix}}$$

General solution: $\boxed{C_1 e^{-t}\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + C_2 e^t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 e^t\begin{pmatrix} t \\ 2 \\ 1 \end{pmatrix}}$