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Math 33B, Lecture 2
Spring 2018
05/21/18
Time Limit: 50 Minutes

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Day \ T.A.	Blaine	Frank	Siting
Tuesday	2A	2C	2E
Thursday	2B	2D	2F

This exam contains 8 pages (including this cover page) and 4 problems. Check to see if any pages are missing.

Instructions

1. Enter your name, SID number, and signature on the top of this page and **cross the box** corresponding to your discussion section.
2. Use a **PEN** to record your final answers.
3. If you need **more space**, use the back of this page and pages 6,8.
4. Calculators, computers, books or notes of any kind are not allowed.
5. Show your work. Unsupported answers will not receive full credit.
6. Good Luck!

Problem	Points	Score
1	18	16
2	18	18
3	20	12
4	14	9
Total:	70	55

1. (18 points) Consider the following differential equation

$$y'' - 2y' + 2y = e^t \sin(t)$$

(a) Find the general solution to the associated homogeneous equation.

(b) Find the general solution to the given inhomogeneous equation.

a) y_H : $y'' - 2y + 2y = 0$

$\lambda^2 - 2\lambda + 2 = 0$

$\lambda = \frac{+2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$y_H = e^{(1+i)t} = e^t e^{it} = e^t(\cos t + i \sin t)$ 9

$\boxed{y_H = c_1 e^t \cos t + c_2 e^t \sin t}$

b) $y'' - 2y' + 2y = e^t \sin t = e^t \cdot \text{Im}(e^{it})$

$y(t) = y_H + y_p$

$z'' - 2z' + 2z = e^{t+it} = e^{(1+i)t}$

$z_p = ae^{t+it} = ae^{(1+i)t}$

$z_p' = a(1+i)e^{(1+i)t}$

$z_p'' = a(1+i)^2 e^{(1+i)t} = a(1+2i-1)e^{(1+i)t} = a(2i)e^{(1+i)t}$

$z_p'' - 2z' + 2z = e^{(1+i)t}$

$2iae^{(1+i)t} - 2(a)(1+i)e^{(1+i)t} + 2ae^{(1+i)t} = e^{(1+i)t}$

$2ia + -(2a)(1+i) + 2a = 1$

$2ia - 2a - 2ia + 2a = 1$

$0 \neq 1$ 7

$z_p = ate^{(1+i)t}$

$z_p' = a(e^{(1+i)t} + (1+i)te^{(1+i)t})$

$z_p'' = a((1+i)e^{(1+i)t} + (1+i)^2 te^{(1+i)t} + (1+i)e^{(1+i)t})$

$= a(2(1+i)e^{(1+i)t} + 2ite^{(1+i)t})$

$a(2(1+i)e^{(1+i)t} + 2ite^{(1+i)t}) - 2a(e^{(1+i)t} + (1+i)te^{(1+i)t}) + 2ate^{(1+i)t} = e^{(1+i)}$

$\cancel{2ae^{(1+i)t}} + 2iae^{(1+i)t} + \cancel{2iate^{(1+i)t}} - \cancel{2ae^{(1+i)t}} - \cancel{2aie^{(1+i)t}}$

$- \cancel{2ate^{(1+i)t}} + \cancel{2ate^{(1+i)t}} = e^{(1+i)t}$

$2ia = 1$

$a = \frac{1}{2i} = \frac{-i}{2}$

$z_p = \frac{-i}{2}te^{(1+i)t}$

$= \frac{-i}{2}te^t(\cos t + i \sin t)$

$y_p = \text{Im}(z_p) = \frac{1}{2}\cancel{te^t}\sin t$ 3

$\boxed{y(t) = c_1 e^t \cos t + c_2 e^t \sin t + \frac{1}{2}\cancel{te^t}\sin t}$

$4t^4 - 12t^4$

$4t^3 \cdot 3t \quad 12t^2 \cdot t^2$

$3t \cdot 2t \quad 3t \cdot 3t^2 \quad 3t \cdot 3t^2 \quad t^2 \cdot 6t$

$9t^3 \quad t^2 \cdot 6t$

2. (18 points) Consider the following differential equation

$2t^2 - 6t^2 + 4t^2 = 0$

$$t^2y'' - 3ty' + 4y = 0 \quad \text{solve for } t > 0$$

$4t^3 - 9t^3 + 6t^3$

$9t^3 \quad 4t^3$

(a) Find two linearly independent solutions of the equation. Show that the solutions are linearly independent.

(b) What is the general solution of the equation. Justify your answer.

a) $r^2 + (p-1)r + q = 0 \quad t^{r_1} = y_1 \quad t^{r_2} = y_2$

$r^2 + (-3-1)r + 4 = r^2 - 4r + 4 = (r-2)^2 = 0$

$r = 2 \ (\times 2)$

$\boxed{y_1 = t^2}$

$W_{y_1,y_2} = \begin{pmatrix} t^2 & y_2 \\ 2t & y_2' \end{pmatrix} = e^{-\int -\frac{3}{t}dt} = e^{3\ln t} = t^3$

$\boxed{y_2 = t^2\ln t + ct^2}$

$t^2y_2' - 2ty_2 = t^3$

$y_2' - \frac{2}{t}y_2 = t$

$\mu = e^{-\int \frac{2}{t}dt} = e^{-2\ln t} = t^{-2}$

linear indp:

$(\mu y_2)' = t \cdot t^{-2} = t^{-1}$

$W = \det\begin{pmatrix} t^2 & t^2\ln t + ct^2 \\ 2t & t + 2t\ln t + 2ct \end{pmatrix}$

$\int(\mu y_2)' = \int t^{-1}dt$

$\mu y_2 = \ln t + C$

$y_2 = t^2\ln t + Ct^2$ take

$= t^3 + 2t^3\ln t + 2ct^3 - 2t^3\ln t - 2ct^3$

$= t^3 \neq 0$

b) $\boxed{y(t) = c_1t^2 + c_2(t^2\ln t + c_3t^2)}$

$y(t) = c_1y_1 + c_2y_2$

3. (20 points) Consider the equation

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$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{\frac{3}{2}}, \quad \text{for } x > 0$$

$\frac{x^{\frac{3}{2}}}{x^2}$ $\frac{3}{2}-\frac{4}{2}$ $x^{-1/2}$

$y'' + \frac{1}{x}y' + \frac{x^2-\frac{1}{4}}{x^2} =$

We are told that the functions $y_1 = x^{-\frac{1}{2}}\sin(x)$ and $y_2 = x^{-\frac{1}{2}}\cos(x)$ are linearly independent solutions of the associated homogeneous equation.

7/12 (a) Find the general solution of the given equation.

0/3 (b) Find a particular solution to the equation satisfying the initial value conditions $y(\pi/2) = 0$, $y'(\pi) = 0$.

5/5 (c) Is it possible to apply the Uniquenss and Existence Theorem for second-order linear equations to the initial value problem in part (b)? Justify your answer.

a) $y(t) = y_H + V_1y_1 + V_2y_2$

$W = \det\begin{pmatrix} x^{-1/2}\sin x & x^{-1/2}\cos x \\ x^{-1/2}\cos x - \frac{1}{2}x^{-3/2}\sin x & -x^{-1/2}\sin x - \frac{1}{2}x^{-3/2}\cos x \end{pmatrix} = x^{-\frac{1}{2}}\sin x\,(x^{-1/2})(\sin x - \frac{1}{2}x^{-1}\cos x)$

$- x^{-1/2}\cos x\,(x^{-1/2})(\cos x - \frac{1}{2}x^{-1}\sin x)$

$x^{-1/2}(\cos x - \frac{1}{2}x^{-1}\sin x)$

$= x^{-1}(\sin^2 x - \cancel{\frac{1}{2}x^{-1}\sin x\cos x}) - x(\cos^2 x - \frac{1}{2}x^{-1}\sin x\cos x)$

$= x^{-1}(\sin^2 x + \cos^2 x) =$ ~~cos2x~~

$f(x) = x^{-1/2}$

$V_1 = -\int \frac{f(x)}{W(x)} y_2\,dx = -\int \frac{x^{-1/2}}{x^{-1}(\sin^2 x - \cos^2 x)} x^{-1/2}\cos x$

$\sin^2 x = 1 - \cos^2 x$

$1 - \cos^2 x - \cos^2 x = 1 - 2\cos^2 x$

$= -\int \frac{\cancel{x^{-1}}\cos x}{\cancel{x^{-1}}(1-2\cos^2 x)}\,dx$

$= \int \frac{\cos x}{\cancel{x}(1-2\cos^2 x)}\,dx$

$= \int \frac{\cos x}{\sin^2 x - \cos^2 x}$

$= \frac{1}{2}\ln|\sin^2 x - \cos^2 x| -$

$V_2 = \int \frac{f(x)}{W(x)} y_1\,dx$

$= \int \frac{x^{-1/2}}{x^{-1}\cos 2x} \cdot x^{-1/2}\sin x\,dx$

$= \int \frac{\sin x}{\cos 2x}$

$h = \cos 2x$

$dh = -\sin 2x$

$= \int \frac{\sin x}{(1-2\cos^2 x)}\,dx$

$h = \cos^2 x$

$dh = -2\cos x\sin x\,dx$

$= \int \frac{\sin x}{x^2 - 2x^2\cos^2 x}\,dx$

$= \int \frac{1}{x^2 - 2x^2 h} \cdot \frac{1}{-2\cos x}\,dx$

(b)?

general solution:

$y(t) = y_H + V_1y_1 + V_2y_2$

$= c_1x^{-1/2}\sin x + c_2x^{-1/2}\cos x + V_1x^{-1/2}\sin x + V_2x^{-1/2}\cos x$

c) You cannot apply the existence + uniqueness theorem because your initial values do not show the same point is continuous and differentiable. $y(\frac{\pi}{2})=0$ shows the function is defined at $x=\frac{\pi}{2}$ while $y'(\pi)=0$ shows the function is defined at $x=\pi$. Since these are different points, the thm can't be applied.

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4. (14 points)

(a) Write the **definition** of what it means for three functions $y_1(t), y_2(t), y_3(t)$ to be linearly independent.

(b) Let y_1, y_2 be solutions to the innhomogeneous linear ODE $y'' - y = f(x)$. We are told that $y_1(0) = 0, y_1'(0) = 2$, $y_2(0) = 2$, $y_2'(0) = 2$, and $y_1(1) = 1$. Find $y_2(1)$. Justify your answer.

a) If $y_1(t), y_2(t), y_3(t)$ are linearly independent, then the only combination c_1, c_2, c_3 that makes the equation $c_1y_1(t) + c_2y_2(t) + c_3y_3(t) = 0$ true is where $c_1 = c_2 = c_3$ are all zero. In other words, no pair of y_1, y_2, y_3 can be scalar multiples of each other ($\frac{y_1}{y_2}, \frac{y_2}{y_3}, \ldots \neq$ constant) ✓ 9

b) $y_1'' + y_1 = f(x)$ $\quad$ $y_2'' - y_2 = f(x)$ $\quad$ 0

$y_1''(0) - y_1(0) = f(0)$ $\quad$ $y_2''(0) - y_2(0) = f(0)$

$y_1''(0) = f(0)$ $\quad$ $y_2''(0) - 2 = f(0)$

$$y_1''(0) = y_2''(0) - 2$$

$$y_1''(1) - y_1(1) = f(1) = y_2''(1) - y_2(1)$$

$$\cancel{y_2''}(1) \cdot 2 - 1 = \cancel{y_2''}(1) - y_2(1)$$

$$\boxed{y_2(1) = 3}$$