Math 33B
Winter 2016
Midterm
02/08/16
Time Limit: 50 Minutes

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This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem. **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit. Formula that you might or might not need:

- The differential form $Pdx+Qdy$ has an integrating factor depending on one of the variables under the following conditions.
If $h = \frac{1}{Q}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$ is a function of x only, then $\mu = e^{\int h(x)dx}$ is an integrating factor.
If $g = \frac{1}{P}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$ is a function of y only, then $\mu = e^{-\int g(x)dx}$ is an integrating factor.

- $e^{i\theta} = \cos\theta + i\sin\theta$

- $\cos(2\theta) = 2\cos^2\theta - 1$

Do not write in the table to the right.

Problem	Points	Score
1	20	20
2	20	18
3	20	19
4	20	13
5	20	7
Total:	100	77

1. (20 points) Solve the equation $\frac{dy}{dx} = \frac{e^{-y}\cos^2 x}{1+y}$ with $y(0) = 1$. Justify your answer.

20

$\int \frac{1+y}{e^{-y}}\, dy = \int \cos^2 x\, dx$

$\int \frac{1}{2}\cos 2x$

$\int (1+y)(e^{y})\, dy = \int \frac{1}{2} + \frac{1}{2}\cos 2x\, dx$

$\frac{1}{2}\left(\frac{1}{2}\right)\int 2$

$u = 1+y \qquad dv = e^{y}\, dy$

$du = dy \qquad v = e^{y}$

$(1+y)(e^{y}) - \int e^{y}\, dy = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

$(1+y)(e^{y}) - e^{y} = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

$ye^{y} = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

$e = \frac{1}{2}(0) + \frac{1}{4}(0) + C$

$C = e$

$ye^{y} = \frac{1}{2}x + \frac{1}{4}\sin 2x + e$

2. (20 points) Solve the equation $\frac{dy}{dx} = \frac{y+xe^{-\frac{y}{x}}}{x}$. Justify your answer.

$y_h = xv$

$x\,dy = (y + xe^{-y/x})\,dx$

$tx = (ty + txe^{-}$

$y_h = xv$

$dy = v\,dx + x\,dv$

$x(v\,dx + x\,dv) - (xv + xe^{-v})\,dx = 0$

$\frac{xv}{x}$

$x^2\,dv - xe^{-v}\,dx = 0$

$x^2\,dv = xe^{-v}\,dx$

18

$\int e^{v}\,dv = \int \frac{1}{x}\,dx$

$e^{v} = \ln|x|$

$v = \ln(\ln(x))$

$y = x(\ln(\ln(x)) + C)$

HOMOGENEOUS

3. (a) (10 points) Solve the equation $y'' + 6y' + 9y = 0$. Here $y = y(t)$ is a function of t. Justify your answer.

10

$\lambda^2 + 6\lambda + 9 = 0$

$(\lambda + 3)(\lambda + 3) = 0$

$\lambda_1 = -3, \ \lambda_2 = -3$

$y_h = C_1 e^{-3t} + C_2 t e^{-3t}$

$At^2 e^{-3t}$

$2A$

(b) (10 points) Solve the equation $y'' + 6y' + 9y = e^{-3t} + t^2$. Justify your answer.

9

$y_p = At^2 e^{-3t} + Bt^2 + Ct + D$

$y_p' = 2Ate^{-3t} - 3At^2e^{-3t} + 2Bt + C$

$y_p'' = 2Ae^{-3t} - 6Ate^{-3t} - 6Ate^{-3t} + 9At^2e^{-3t} + 2B$

$= 9At^2e^{-3t} - 12Ate^{-3t} + 2Ae^{-3t} + 2B$

$\cancel{9At^2e^{-3t}} - \cancel{12Ate^{-3t}} + 2Ae^{-3t} + 2B + \cancel{12Ate^{-3t}} - \cancel{18At^2e^{-3t}} + \boxed{12Bt} + 6C$

$+ \cancel{9At^2e^{-3t}} + 9Bt^2 + \boxed{9Ct} + 9D$

$2Ae^{-3t} + 9Bt^2 + (12B + 9C)t + (2B + 6C + 9D) = e^{-3t} + t^2$

$A = 1/2, \ B = 1/9$ $\quad \frac{12}{9} + 9c = 0 \quad c = -\frac{1}{12}$ $\quad 12B + 9C = 0$ $\quad 12/9 + 9C = 0$ $\quad 9c = -12/9$ $\quad c = -12/81$

$\boxed{y = C_1 e^{-3t} + C_2 t e^{-3t} + 1/2\, t^2 e^{-3t} + 1/9\, t^2 - \frac{12}{81}t + \frac{56}{9(81)}}$

$D = \frac{56}{9(81)}$

$\frac{56}{81} = 9D$

$\frac{2}{9} - \frac{72}{81} + 9D$ $\quad \frac{18}{81} - \frac{72}{81} + 9D$

$72 - 18 = 56$

4. Let $f(t) = te^{2t}$ and $g(t) = t^2$ be two functions of t.

(a) (4 points) Compute the Wronskian of f and g.

$$\det\left(\begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}\right) \qquad f(t) = te^{2t} \qquad g(t) = t^2$$

$$f'(t) = e^{2t} + 2te^{2t} \qquad g'(t) = 2t$$

$$(te^{2t})(2t) - (e^{2t} + 2te^{2t})(t^2)$$

$$2t^2e^{2t} - t^2e^{2t} - 2t^3e^{2t}$$

$$t^2e^{2t} - 2t^3e^{2t}$$

$$W = t^2e^{2t}(1-2t)$$

(b) (8 points) Show that f are g are linearly independent functions of t.

f and g are linearly independent functions of t because the Wronskian is not equal to 0. If the Wronskian were equal to 0, then f and g would be linearly dependent.

(c) (8 points) Show that there do not exist differentiable functions $p(t)$ and $q(t)$ such that $y(t) = c_1 f(t) + c_2 g(t)$ is the general solution to $y'' + p(t)y' + q(t)y = 0$, where c_1 and c_2 are arbitrary constants.

$$y(t) = c_1 te^{2t} + c_2 t^2$$

$$y'(t) = C_1 e^{2t} + C_1 2te^{2t} + 2C_2 t$$

$$y''(t) = 2C_1 e^{2t} + C_1(e^{2t} + 2tC_1 e^{2t}) + 2C_2$$

$$= 3C_1 e^{2t} + 2tC_1^2 e^{2t} + 2C_2$$

$$\underline{3C_1 e^{2t}} + 2tC_1^2 e^{2t} + 2C_2 + p(t)(\underline{C_1 e^{2t}} + C_1 2te^{2t} + 2C_2 t) + q(t)(C_1 te^{2t} + C_2 t^2) = 0$$

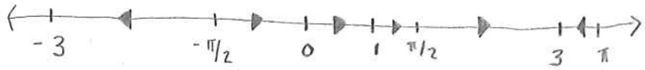
$$p(t)((\underset{\substack{\uparrow \\ 0}}{C_1 e^{2t}} + C_1 2te^{2t} + 2C_2 t)$$

Therefore, no differentiable functions since $y'(t) = 0$

5. $\frac{dy}{dt} = 3y^{\frac{2}{3}}(y-1)(y^2-9)\cos y$ $\qquad \cos y = 0 \qquad y = \pi/2$

(a) (10 points) Find all the constant solutions that are (asymptotically) stable. Justify your answer.

$y = 0, 1, +3, -3, \pi/2, -\pi/2$ $\qquad$ 1.5



3 is stable.

4/10

+ - - +

$3\left(\frac{2}{3}\right)y^{-1/3}$ $\qquad \frac{2}{\sqrt[3]{y}}$

(b) (10 points) Are the solutions to the above equation together with the initial condition $y(0) = 1$ unique? Justify your answer.

$\frac{dy}{dt} = 3y^{2/3}\cos y\,(y^3 - y^2 - 9y + 9)$ $\qquad \frac{18}{3}\frac{1}{\sqrt[3]{y}}$

$= 3\cos y\,(y^{11/3} - y^{8/3} - 9y^{5/3} + 9y^{2/3})$

$y'' = -3\sin y\,(y^{11/3} - y^{8/3} - 9y^{5/3} + 9y^{2/3}) + 3\cos y\,(\frac{11}{3}y^{8/3} - \frac{8}{3}y^{5/3} - \frac{45}{3}y^{2/3} + \frac{18}{3}y^{-1/3})$

not continuous and differentiable on y''

uniqueness theorem not guaranteed.

Not unique because cos y

3/10

yes unique

if not unique,
then all would go thru y(0)