1. (20 points) Solve the equation $\frac{dy}{dx} = \frac{e^{-y}\cos^2 x}{1+y}$ with $y(0) = 1$. Justify your answer.

14

$\cancel{\int e^y(1+y)\,dy = \int \cos^2 x\,dx}$

$u\,dv = uv - \int v\,du$

$e^y(1+y)\,dy = \cos^2 x\,dx$

$2e^y(1+y)\,dy = 2\cos^2 x\,dx$

$(2e^y(1+y)+1)\,dy = (2\cos^2 x+1)\,dx$

$\int (2e^y + 2ye^y + 1)\,dy = \int \cos(2x)\,dx$

$2(ye^y)$ $u=y$ $du=1$ $dv=e^y$ $v=e^y$

$ye^y - \int e^y$

$ye^y - e^y$

$(y-1)e^y$

$2e^y + 2(y-1)e^y + y = \frac{1}{2}\sin(2x) + C$

$2e^y + 2ye^y - 2e^y + y = \frac{1}{2}\sin(2x) + C$

$2ye^y + y = \frac{1}{2}\sin(2x) + C$

$y(2e^y+1) = \frac{1}{2}\sin(2x) + C$

$1(2e+1) = \frac{1}{2}\sin(0) + C$ $F\ y(0)=1$

$2e+1 = C$

$$\boxed{y(2e^y+1) = \frac{1}{2}\sin(2x) + 2e + 1}$$

14

2. (20 points) Solve the equation $\frac{dy}{dx} = \frac{y+xe^{-\frac{y}{x}}}{x}$. Justify your answer.

6

$\cancel{y' = \frac{y}{x} + e^{-y/x}}$

$dy(x) = (y + xe^{-y/(x)})\,dx$

$(y + xe^{-y/x})\,dx - x\,dy = 0$

$\frac{dy}{dx} = \frac{y}{x} + e^{-y/x}$

$= \frac{y}{x} + \frac{1}{e^{y/x}}$

homog in degree 1

$(xu + xe^{-xu/x})\,dx - x\,dy = 0$

$x(u + e^{-u})\,dx - x\,dy = 0$

$(u + e^{-u})\,dx - 1(x\,du + u\,dx) = 0$

$(u + e^{-u})\,dx - x\,du - u\,dx = 0$

$-x\,du = (u - u - e^{-u})\,dx$

$y = \frac{1}{y + xe^{-y/x}}\left(1 - e^{-y/x} + 1\right)$

$y = \frac{1}{-x}\left(1 - e^{-y/x} + 1\right)$

4. Let $f(t) = te^{2t}$ and $g(t) = t^2$ be two functions of t.

(a) (4 points) Compute the Wronskian of f and g.

$te^{2t} \cdot 2t - (e^{2t} + 2te^{2t})t^2$

$2t^2e^{2t} - t^2e^{2t} + 2t^3e^{2t}$

$2t^3e^{2t} + t^2e^{2t}$

$\boxed{t^2e^{2t}(2t+1)}$

X 3

(b) (8 points) Show that f are g are linearly independent functions of t.

If $\frac{f(t)}{g(t)} \neq c$, then f & g are LI.

$\frac{te^{2t}}{t^2} = \frac{e^{2t}}{t} \neq c$

$\therefore$ L.I.

8

(c) (8 points) Show that there do not exist differentiable functions $p(t)$ and $q(t)$ such that $y(t) = c_1f(t) + c_2g(t)$ is the general solution to $y'' + p(t)y' + q(t)y = 0$, where c_1 and c_2 are arbitrary constants.

In all solutions of y, $f(t)$ & $g(t)$ must be of the form $te^{\lambda t}$, $e^{\lambda t}$, or $e^{\alpha t}\cos\beta t$ / $e^{\alpha t}\sin\beta t$.

$g(t)$ is in none of these forms.

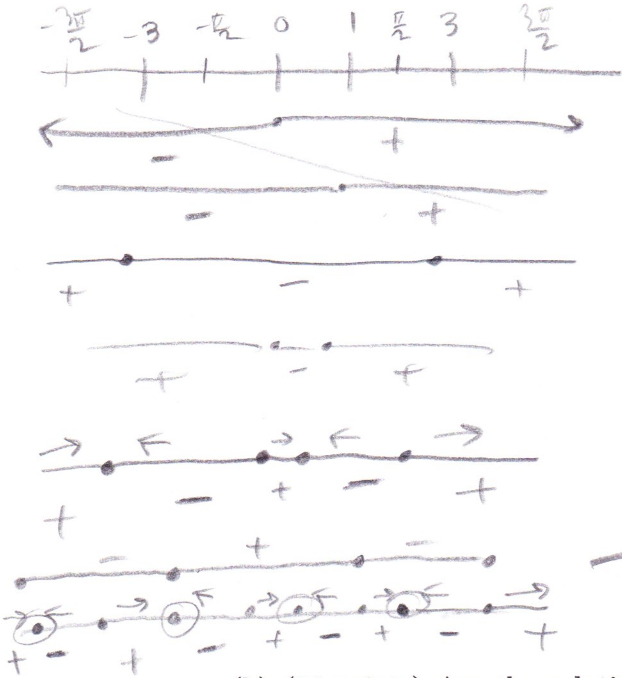
X 1

5. $\frac{dy}{dt} = 3y^{\frac{2}{3}}(y-1)(y^2-9)\cos y$

(a) (10 points) Find all the constant solutions that are (asymptotically) stable. Justify your answer.

$y = 0, 1, 3, -3, n\frac{\pi}{2}, n\frac{3\pi}{2}$

$\boxed{y = 1, \frac{-\pi}{2}, -n\frac{3\pi}{2}, \frac{n\pi}{2}}$



$-n\frac{3\pi}{2}, n\frac{\pi}{2}$

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$\longrightarrow$ cos y alts between stable & non stable in neg dir it's $\frac{3\pi}{2}$, pos $\rightarrow \frac{\pi}{2}$.

(b) (10 points) Are the solutions to the above equation together with the initial condition $y(0) = 1$ unique? Justify your answer.

$y' = \int f(t)\, 3y^{2/3}$

$= \frac{d}{c} f(t)\, 3y^{2/3} + \int f \cdot 2 \cdot y^{-1/3}$

ok term $\hookrightarrow \frac{1}{y^{1/3}} \rightarrow$ not continuous @ 0.

$\therefore$ not unique because it cannot satisfy uniqueness theorem conditions @ 0.

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