Math 33B
Winter 2016
Midterm
02/08/16
Time Limit: 50 Minutes

Name (Print): [illegible]

TA name [illegible] section number

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem. **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit. Formula that you might or might not need:

- The differential form $Pdx+Qdy$ has an integrating factor depending on one of the variables under the following conditions.
  If $h = \frac{1}{Q}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$ is a function of x only, then $\mu = e^{\int h(x)dx}$ is an integrating factor.
  If $g = \frac{1}{P}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$ is a function of y only, then $\mu = e^{-\int g(x)dx}$ is an integrating factor.
- $e^{i\theta} = \cos\theta + i\sin\theta$
- $\cos(2\theta) = 2\cos^2\theta - 1$

Do not write in the table to the right.

Problem	Points	Score
1	20	20
2	20	20
3	20	13
4	20	12
5	20	9
Total:	100	74

AL

simplify?

1. (20 points) Solve the equation $\frac{dy}{dx} = \frac{e^{-y}\cos^2 x}{1+y}$ with $y(0) = 1$. Justify your answer.

$y \neq -1$

separable

20

$\frac{dy}{dx}(1+y) = e^{-y}\cos^2 x$

$dy\,(1+y)\,e^{y} = \cos^2 x\,dx$

$\int e^{y}(1+y)\,dy = \int \cos^2 x\,dx$

$u = 1+y \quad dv = e^{y}dy$

$du = dy \quad v = e^{y}$

$\int \frac{1+\cos(2x)}{2}dx$

$\frac{x}{2} + \frac{\sin(2x)}{4}$

$e^{y}(1+y) - \int e^{y}\,dy$

$e^{y}(1+y) - e^{y} + C = \frac{x}{2} + \frac{\sin(2x)}{4}$

$e^{y}(1+y-1) = \frac{x}{2} + \frac{\sin(2x)}{4} + C$

$e^{y}y = \frac{x}{2} + \frac{\sin(2x)}{4} + C \qquad y(0) = 1$

$e^{1}(1) = 0 + 0 + C$

$e = C$

$e^{y}y = \frac{x}{2} + \frac{\sin(2x)}{4} + e$

$(y \neq -1)$

2. (20 points) Solve the equation $\frac{dy}{dx} = \frac{y+xe^{-\frac{y}{x}}}{x}$. Justify your answer. $(x \neq 0)$

$$\frac{dy}{dx} = \frac{y + xe^{-\frac{y}{x}}}{x}$$

homog? $\frac{ty + txe^{-\frac{y}{x}}}{tx} = \frac{y + xe^{-\frac{y}{x}}}{x}$ yes. homog of deg 0

$$y = vx \quad \frac{dy}{dx} = \frac{dv}{dx}x + v$$

$$\frac{dv}{dx}x + v = \frac{vx + xe^{-(vx/x)}}{x} = \frac{vx + xe^{-v}}{x} = v + e^{-v}$$

$$\frac{dv}{dx}x + \cancel{v} = \cancel{v} + e^{-v}$$

$$\frac{dv}{dx}x = e^{-v}$$

$$\int e^{v}\,dv = \int \frac{1}{x}\,dx$$

$$e^{v} = \ln|x| + C$$

$$v = \ln(\ln|x| + C) \qquad v = \frac{y}{x}$$

$$\frac{y}{x} = \ln(\ln|x| + C)$$

$$y = x\ln(\ln|x| + C)$$

✓

20

AL

3. (a) (10 points) Solve the equation $y'' + 6y' + 9y = 0$. Here $y = y(t)$ is a function of t. Justify your answer. 2nd order linear DE

Characteristic eqn: $\lambda^2 + 6\lambda + 9 = 0$

$(\lambda+3)^2 = 0 \quad \lambda = -3 \rightarrow c_1 e^{-3t}$ is one solution to the DE ~~part of the solution~~

10

Try $y_p = te^{-3t}$

$y_p' = t(-3e^{-3t}) + e^{-3t} = e^{-3t}(-3t+1)$

$y_p'' = (-3t+1)(-3e^{-3t}) + e^{-3t}(-3) = (-3e^{-3t})(-3t+1+1)$

$= -3e^{-3t}(-3t+2)$

$-3e^{-3t}(-3t+2) + 6(e^{-3t}(-3t+1)) + 9te^{-3t} \stackrel{?}{=} 0$

$-3(-3t+2) + 6(-3t+1) + 9t = 0$

$+9t - 6 + -18t + 6 + 9t = 0$

$-6+6=0$

$0=0$ ✓ $\Rightarrow \boxed{y = c_1 e^{-3t} + c_2 te^{-3t}}$

need a y_p

(b) (10 points) Solve the equation $y'' + 6y' + 9y = e^{-3t} + t^2$. Justify your answer.

3

Solve homog eqn $y'' + 6y' + 9y = 0$ (a)

$y = c_1 e^{-3t} + c_2 te^{-3t}$

① $y'' + 6y' + 9y = t^2$ guess $y_p = at^2 + bt + c$

② $y'' + 6y' + 9y = e^{-3t} \rightarrow$ guess $y_p = Ae^{-3t}$

But that's a soln to homog. eqn, so guess At^2e^{-3t}

Remember you can break up forcing term

Find a particular soln to this eqn. Guess $y_p = ae^{-3t} + bt^2$

$y_p = ae^{-3t} + bt^2$

$y_p' = -3ae^{-3t} + 2bt$

$y_p'' = 9ae^{-3t} + 2b$

$(9ae^{-3t} + 2b) + 6(-3ae^{-3t} + 2bt) + 9(ae^{-3t} + bt^2)$

$\stackrel{?}{=} e^{-3t} + t^2$

$9ae^{-3t} - 18ae^{-3t} + 9ae^{-3t} + 2b + 12bt + 9bt^2$

$2b + 12bt + 9bt^2 = e^{-3t} + t^2 \quad \forall t$ didn't work

$y = v_1 y_1 + v_2 y_2$

~~1st~~ $v_1 = \int \frac{-y_2 f}{W}$ $\quad v_2 = \int \frac{-y_1 f}{W}$

$y_1 = e^{-3t}$ $\quad y_2 = te^{-3t}$

$y_1' = -3e^{-3t}$ $\quad y_2' = e^{-3t} + t(-3e^{-3t})$

$= e^{-3t}(-3t+1)$

$W = e^{-6t}(-3t+1) - -3te^{-6t}$

$W = e^{-6t}(-3t+1) + 3te^{-6t}$

$= -3te^{-6t} + 3te^{-3t} + e^{-6t} = W$

$v_2 = \int e^{-3t}($

$v_1 = \int \frac{-te^{-3t}(e^{-3t}+t^2)}{e^{-6t}} dt = \int -t$

AL

4. Let $f(t) = te^{2t}$ and $g(t) = t^2$ be two functions of t.

(a) (4 points) Compute the Wronskian of f and g.

$W = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$

$W = f(t)g'(t) - f'(t)g(t)$

$te^{2t}(2t) - (e^{2t} + t(2e^{2t}))t^2$

$2t^2e^{2t} - (e^{2t} + 2te^{2t})t^2$

$2t^2e^{2t} - t^2e^{2t} - 2t^3e^{2t}$

$W = t^2e^{2t} - 2t^3e^{2t}$

✓

(b) (8 points) Show that f are g are linearly independent functions of t.

~~$W(t) = 0 \; \forall t \iff f$ and g are linearly independent. From (a), $W = t^2e^{2t} - 2t^3e^{2t}$~~

Also, To show f and g are lin. ind., assume they are lin. dependent

then $f = cg$ where c is a constant

$te^{2t} = t^2c \quad c = \frac{te^{2t}}{t^2} = \frac{e^{2t}}{t}$, which is not a constant.

$\Rightarrow f$ and g are not lin. dependent

$\Rightarrow f$ and g are lin. ind.

✓

(c) (8 points) Show that there do not exist differentiable functions $p(t)$ and $q(t)$ such that $y(t) = c_1f(t) + c_2g(t)$ is the general solution to $y'' + p(t)y' + q(t)y = 0$, where c_1 and c_2 are arbitrary constants.

Suppose $\exists$ such a $p(t)$ and $q(t)$

then $y(t) = c_1f(t) + c_2g(t)$ is a soln

but ~~if~~ ~~a~~ that's the case, ()

then $W = 0 \; \forall t$ or $W \neq 0 \; \forall t$

By (a) $W(0) = 0$ and $W(1) \neq 0$

~~th~~ $\Rightarrow$ contradiction

$\Rightarrow$ there does not exist such a $p(t)$ and $q(t)$

A L

5. $\frac{dy}{dt} = 3y^{\frac{2}{3}}(y-1)(y^2-9)\cos y$

(a) (10 points) Find all the constant solutions that are (asymptotically) stable. Justify your answer.

Equilibrium sol'ns: $\left(\text{at } \frac{dy}{dt} = 0\right) \to y = 0, 1, 3, -3, \left[\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\right]$

Thm: If $\frac{d}{dy}\left(3y^{2/3}(y-1)(y^2-9)\cos y\right) < 0$ at an equilibrium point, then that point is asymptotically stable (from the book)

$$\frac{dy}{dt} = \left(3y^{\frac{5}{3}} - 3y^{\frac{2}{3}}\right)(y^2-9)\cos y = \left(3y^{\frac{11}{3}} - 27y^{\frac{5}{3}} - 3y^{\frac{8}{3}} + 27y^{\frac{2}{3}}\right)\cos y$$

$$\frac{d}{dy}\left(\frac{dy}{dt}\right) = \left(\frac{11}{\cancel{3}}\cdot\cancel{3}y^{\frac{8}{3}} - \cancel{27}^{9}\cdot\frac{5}{\cancel{3}}y^{\frac{2}{3}} - \frac{8}{\cancel{3}}\cdot\cancel{3}y^{\frac{5}{3}} + \frac{2}{\cancel{3}}\cdot\cancel{27}^{9}y^{-\frac{1}{3}}\right)\cos y$$

$$-(\sin y)\left(3y^{\frac{11}{3}} - 3y^{\frac{8}{3}} - 27y^{\frac{5}{3}} + 27y^{\frac{2}{3}}\right)$$

Try phase line.

$$= \left(11y^{\frac{8}{3}} - 45y^{\frac{2}{3}} - 8y^{\frac{5}{3}} + 18y^{-\frac{1}{3}}\right)\cos y - \left(3y^{\frac{11}{3}} - 3y^{\frac{8}{3}} - 27y^{\frac{5}{3}} + 27y^{\frac{2}{3}}\right)\sin y$$

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(b) (10 points) Are the solutions to the above equation together with the initial condition $y(0) = 1$ unique? Justify your answer.

They are unique if $\frac{dy}{dt}$ and $\frac{d}{dy}\left(\frac{dy}{dt}\right)$ are both continuous around $t=0$

By (a), $\frac{d}{dy}\left(\frac{dy}{dt}\right)$ contains $18y^{-\frac{1}{3}} = \frac{18}{\sqrt[3]{y}}$, which is not continuous at $y = $ ~~0~~

$\Rightarrow$ solution is not unique

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