7. (10 points) (a) Verify that $y_1 = t$ and $y_2 = t^2$ form a fundamental set of solutions to the homogeneous equation

$$t^2y'' - 2ty' + 2y = 0 \tag{6}$$

4

on the interval (2, 10).

10

$w(t) = \det\begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} = 2t^2 - t^2 = t^2$ and t^2 is never 0 on the interval (2, 10)

$\Rightarrow$ y_1 and y_2 are linearly independent

check y_1: $y_1' = 1$ $y_1'' = 0 \Rightarrow t^2(0) - 2t(1) + 2(t) = 2t - 2t = 0$ ✓

check y_2: $y_2' = 2t$ $y_2'' = 2 \Rightarrow t^2(2) - 2t(2t) + 2t^2 = 4t^2 - 4t^2 = 0$ ✓

$\Rightarrow$ Based on $w(t)$ and 2 checks, y_1 and y_2 form fundamental set of solutions to homogeneous eq.

(b) Use variation of parameters to find a particular solution to

6

$$t^2y'' - 2ty' + 2y = \frac{2t^3}{1+t^2}. \tag{7}$$

Box your answer *Hint:* you should have no trouble evaluating the integrals.

(7) is equal to $y'' - 2t^{-1}y' + 2t^{-2}y = \frac{2t}{(1+t^2)}$

$y_p = v_1 y_1 + v_2 y_2 = v_1 \cdot (t) + v_2 \cdot (t^2)$

$v_1 = \int \frac{-t^2\left(\frac{2t}{1+t^2}\right)}{t^2} dt = -\int \frac{2t}{1+t^2} dt$ $u = 1+t^2$ $du = 2t\,dt$

$= -\int \frac{du}{u} = -\ln|1+t^2| = -\ln(1+t^2)$

$v_2 = \int \frac{t\left(\frac{2t}{1+t^2}\right)}{t^2} dt = \int \frac{2\cancel{t}}{\cancel{t}(1+t^2)} dt = \int \frac{2}{1+t^2} dt = 2\arctan(t)$

$$\boxed{y_p = -\ln(1+t^2)\,t + 2\arctan(t)\,t^2}$$

4. (4 points) True or false: there exists a differential equation of the form $y'' + p(t)y' + q(t)y = 0$, with p and q continuous functions on $(-\infty, \infty)$, such that

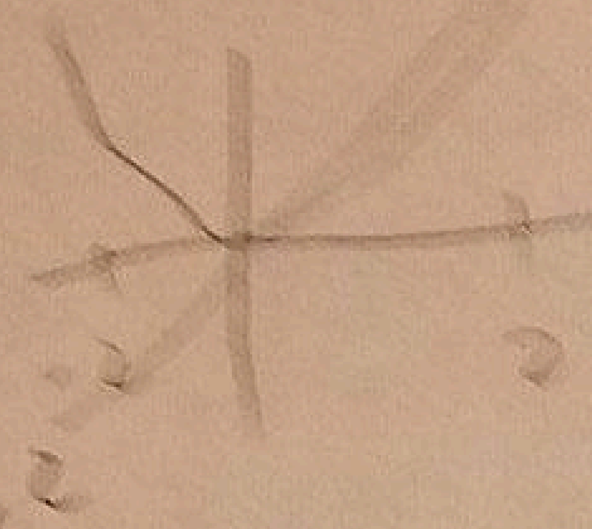
$$y_1 = t \quad \text{and} \quad y_2 = t^2$$

form a fundamental set of solutions in the interval $(-2, 2)$.

(4)

A. True

B. False



$t \quad t^2$

$1 \quad 2t$

$2t^2 - t^2$

$\boxed{B}$

5. (4 points) Suppose that y_1 and y_2 are linearly independent solutions to the differential equation $y'' + p(t)y' + q(t)y = 0$. Which of the following pairs is linearly dependent?

A. $y_3 = -y_2$ and $y_4 = 5y_1 + 6y_2$

B. $y_3 = -2y_1 + 3y_2$ and $y_4 = 8y_1 - 12y_2$

C. $y_3 = 5y_1 + 4y_2$ and $y_4 = 3y_1 - 2y_2$

D. $y_3 = y_1 - y_2$ and $y_4 = y_1$

E. $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$

$y_1 \quad y_2$

$y_1' \quad y_2'$

$-y_2 \quad 5y_1$

$-y_2'$

$y_1 + y_2 \quad y_1 - y_2$

$y_1' + y_2' \quad y_1' - y_2'$

$(y_1 + y_2)(y_1' - y_2') - (y_1' + y_2')(y_1 - y_2)$

$y_1 y_1' - y_1 y_2' + y_2 y_1' - y_2 y_2'$

$- y_1' y_1 - y_1 y_2' + y_1' y_2 + y_2' y_2$

$y_1' y_2 - y_1 y_2'$

$\boxed{E}$

3. (4 points) Use the form $y_p = (at+b)e^{*t}$ to find a particular solution of the equation

$$y'' + 4y' + 2y = (8t-2)e^{-4t}. \qquad (3)$$

A. $y_p = 8e^{(-2+\sqrt{2})t} - 2e^{(-2-\sqrt{2})t}$

B. $y_p = 4e^{(-2+\sqrt{2})t} + 7e^{(-2-\sqrt{2})t}$

C. $y_p = 7e^{-4t} + 4te^{-4t}$

D. $y_p = -2e^{-4t} + 8te^{-4t}$

E. $y_p = -3te^{-4t}$

try $y = (at+b)e^{-4t}$

$y' = -4(at+b)e^{-4t} + e^{-4t}(a)$

$y'' = 16(at+b)e^{-4t} + e^{-4t}(-4a) + (-4a)e^{-4t}$

$-16(at+b)e^{-4t} + 4(e^{-4t})a + 2(at+b)e^{-4t}$

$(-4a)e^{-4t} + 2ate^{-4t} + 2be^{-4t} = (8t-2)e^{-4t}$

$(-4a + 2at + 2b)e^{-4t} = (8t-2)e^{-4t}$

$2a = 8 \Rightarrow a = 4$

$2b - 4a = -2 \Rightarrow$

$2b = 14$

$b = 7$

$(4t+7)e^{-4t}$

C

8. (10 points) The population y of the passenger pidgeon[1] is modeled by the differential equation

$$\frac{dy}{dt} = -ry\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right), \tag{8}$$

where r, T, and K are constants with $r > 0$ and $0 < T < K$. You may assume that $y \geq 0$. $y<0$

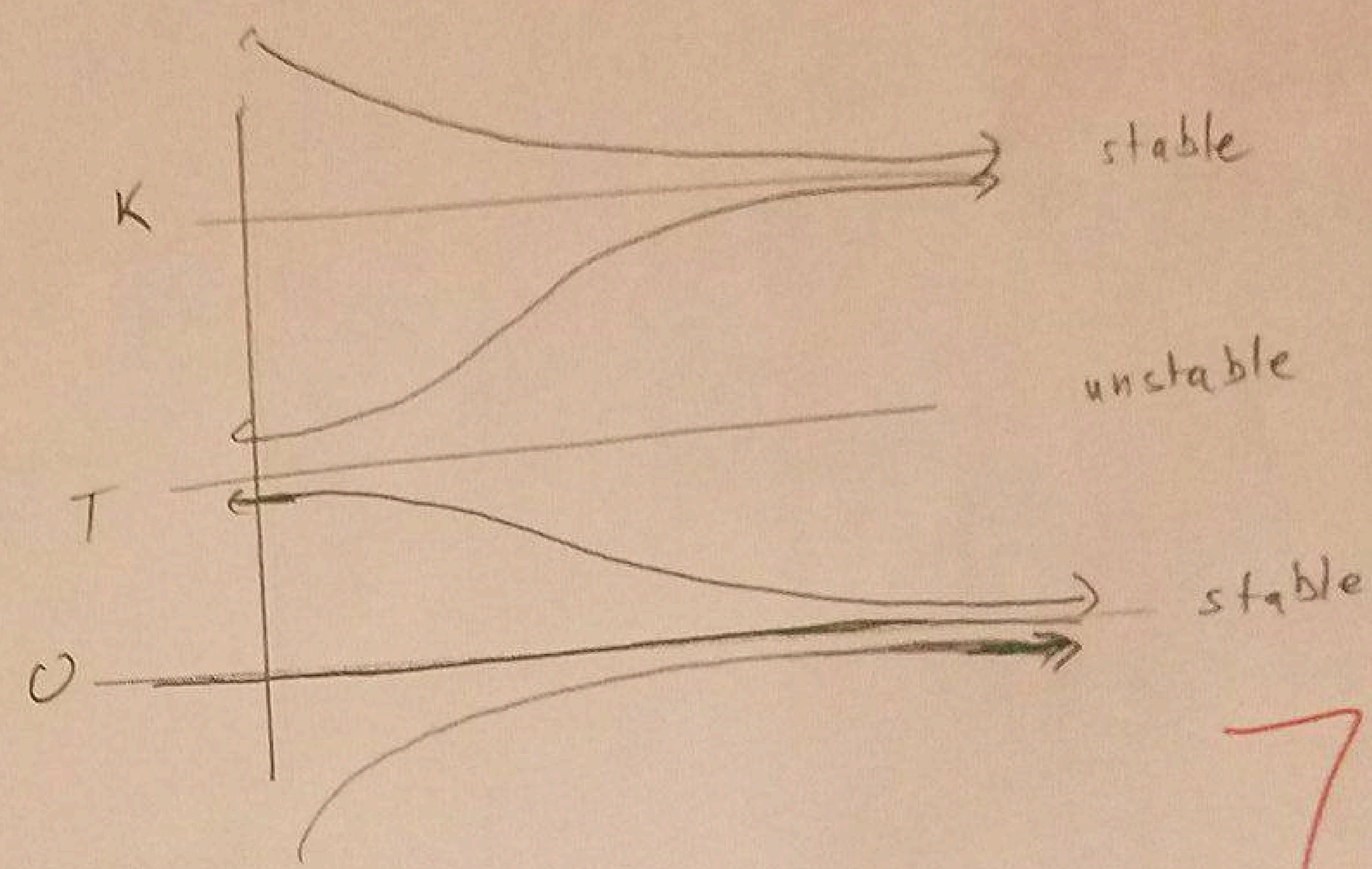
(a) Find the equilibrium points and classify each as either stable or unstable. Sketch the equilibrium solutions in the ty-plane. These equilibrium solutions divide the ty-plane into regions. Sketch at least one solution trajectory in each of these regions.

eq points when $\frac{dy}{dt} = 0$

$\to$ $y = 0$ $(ry = 0)$ stable

$y = T$ $\left(1 - \frac{y}{T} = 0\right)$ unstable

$y = K$ $\left(1 - \frac{y}{K} = 0\right)$ stable



7

(b) Suppose that $K = 2T$ in the equation above. For which values of y in $[0, K]$ is the rate of change of the population at its maximum/minimum? There is no partial credit on this problem and you don't need to show work (so you may do this on scratch paper and write your answers below). $\boxed{\text{Box your answers}}$

~~$\frac{dy}{dt}$ is at minimum at $y = 0$~~

~~$\frac{dy}{dt}$ is at maximum at $y = K = 2T$~~

0

[1] In the mid nineteenth century, the passenger pidgeon was heavily hunted for food and sport, which drastically reduced its numbers. Apparently the passenger pidgeon could only breed successfully when present in a large concentration (i.e. more than T). By the 1880s the population had declined to below the threshold T, after which the population rapidly declined to extinction. This event was one of the early factors contributing to a concern for conservation in the United States.

7

$\begin{array}{r} 41 \\ -24 \\ \hline 17 \end{array}$

$\lambda^2 + 4\lambda + 5 = 0$

$\frac{-4 \pm \sqrt{16 \cdot 20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm$

Part II: Free Response. Write up a full solution for each problem. Unless otherwise indicated, a correct answer with an incomplete or incorrect solution will not receive full credit.

6. (10 points) Find the general solution to the differential equation

$$y'' + 4y' + 5y = 10 - 17e^{-6t}. \qquad (5)$$

Box your answer

(1) $y'' + 4y' + 5y = 10$ (2) $y'' + 4y' + 5y = -17e^{-6t}$

* Solve homogeneous eq (1):

try $y = a \Rightarrow y' = 0 \quad y'' = 0 \Rightarrow 0 + 4(0) + 5(a) = 10$

$\Rightarrow a = 2 \Rightarrow y_{P,1} = 2$

* Solve homo eq (2):

try $y = ae^{-6t} \Rightarrow y' = -6ae^{-6t} \quad y'' = 36ae^{-6t}$

$\Rightarrow 36ae^{-6t} + (-24)ae^{-6t} + 5ae^{-6t} = -17e^{-6t}$

$17ae^{-6t} = -17e^{-6t} \Rightarrow a = -1$

$y_{P,2} = -e^{-6t}$

* Solve homo eq general: $y'' + 4y' + 5y = 0$

$\lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

$y_h = Ae^{-2t}\cos t + Be^{-2t}\sin t$

$\Rightarrow$ general sol to diff eq (5) is $y = y_{P,1} + y_{P,2} + y_h$

$$\boxed{y = 2 + (-e^{-6t}) + Ae^{-2t}\cos t + Be^{-2t}\sin t}$$

Part I: Multiple choice. Please write your answers (A, B, C, ...) in the boxes on the right.

1. (4 points) Consider the differential equation

$$ay'' + by' + cy = 0, \qquad (1)$$

where a, b, c are constants and $a \neq 0$. Which of the following is true about the form of the general solution? (A and B are arbitrary constants below)

A. If $b^2 - 4ac > 0$ then $y = A\cos\lambda t + B\sin\lambda t$ for some real constant λ.

B. If $b^2 - 4ac > 0$ then $y = A\cos\lambda t + B\,t\cos\lambda t$ for some real constant λ.

C. If $b^2 - 4ac = 0$ then $y = Ae^{-bt/2a} + B\,te^{-bt/2a}$.

D. If $b^2 - 4ac = 0$ then $y = Ae^{at}\cos bt + Be^{at}\sin bt$.

E. If $b^2 - 4ac < 0$ then $y = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ for some real constants λ_1, λ_2.

F. If $b^2 - 4ac < 0$ then $y = Ae^{\lambda t} + B\,te^{\lambda t}$ for some real constant λ.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{-b}{2a}$$

$\boxed{C}$

2. (4 points) Find the general solution to the equation

$$y'' + 3y' - 10y = 0. \qquad (2)$$

$\lambda \quad -2$
$\lambda \quad +5$

$\lambda^2 + 3 - 10 =$

$(\lambda + 5)(\lambda - 2)$

$\lambda = -5$
$\lambda = 2$

$Ae^{-5t} + Be^{2t}$

A. $y = Ae^{-5it} + Be^{2it}$

B. $y = A\cos(\sqrt{10}\,t) + B\sin(\sqrt{10}\,t)$

C. $y = Ae^{-5t} + B\,te^{-5t}$

D. $y = Ae^{5t} + Be^{-2t}$

E. $y = Ae^{-2t} + Be^{-2t}\ln(5t)$

$\lambda^2 - 3\lambda - 10 = 0$

$(\lambda - 5)(\lambda + 2) = 0$

$\lambda = 5$

$\lambda = -2$

$e^{-1} e^{5t}$

$\cancel{A} e^{-1} e^{5t}$

$\boxed{D}$