MATH 33B Exam 1 [v.1a] Fall 2017

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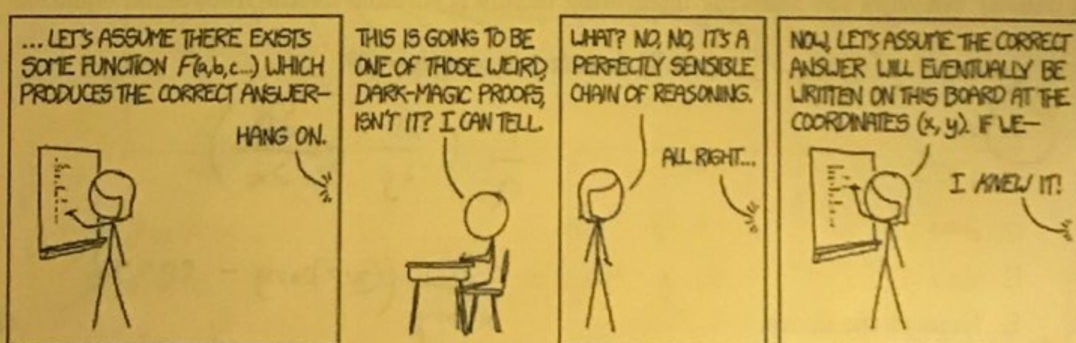
Section	1	B

- Do not open this exam packet until you are told that you may begin.
- Turn off all electronic devices and and put away all items except for a pen/pencil and an eraser.
- Remove hats and sunglasses.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time.
- If you have a question, raise your hand and one of the proctors will come to you. We will not answer any mathematical questions except possibly to clarify the wording of a problem.
- Quit working and close this packet when you are told to stop.



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Points:	7	4	7	7	6	5	8	6	50
Score:	7	4	4	7	6	5	8	6	47

3

Part I: Multiple choice. Please write your answers (A, B, C, ...) in the boxes on the right.

1. (3 points) Determine values of the constants a and b that make the differential equation exact.

$$(3x^2y + ax^5y^3)\,dx + (bx^3 + x^6y^2)\,dy = 0 \qquad (1)$$

$3x^2y + 2x^5y^3$ $\to 3x^2 + 6x^5y^2$

$x^3 + x^6y^2$ $\to 3x^2 + 6x^5y^2$

A. $a = 2, b = -1$

B. $a = 1, b = 2$

C. $a = 2, b = 1$

D. $a = -1, b = 2$

E. None of the above.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$3x^2 + 3ax^5y^2 = 3bx^2 + 6y^2x^5$$

$$\cancel{x^2(3 + 3ax^3y^2) = x^2(3b + 6y^2x^3)}$$

$$1 + ax^3y^2 = b + 2x^3y^2$$

$$\cancel{1 + ax^3y^2 = b + 2y^2x^3}$$

$$1 - b = (2-a)x^3y^2$$

$$\cancel{1 - b = (2-a)}y^2x^3$$

C

2. (4 points) Which of the following integrating factors is suitable for the differential equation

$$(x+2)\sin y\,dx + x\cos y\,dy = 0? \qquad (2)$$

A. xe^x

B. $1 + \frac{1}{x}$

C. $e^{\cos x}$

D. $\sin x$

E. None of the above.

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= \frac{1}{x\cos y}\left((x+2)\cos y - \cos y\right)$$

$$= \frac{x\cos y + 2\cos y - \cos y}{x\cos y} = \frac{\cos y\,(x+2-1)}{x\cos y}$$

$$= \frac{x+1}{x}$$

$$\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \frac{1}{(x+2)\sin y}\left(\cos y - (x+2)\cos y\right)$$

$$= \frac{\cos y\,(1-(x+2))}{x+2\,(\sin y)}$$

$$e^{\int \frac{x+1}{x}}$$

$$= e^{\int 1 + \frac{1}{x}}$$

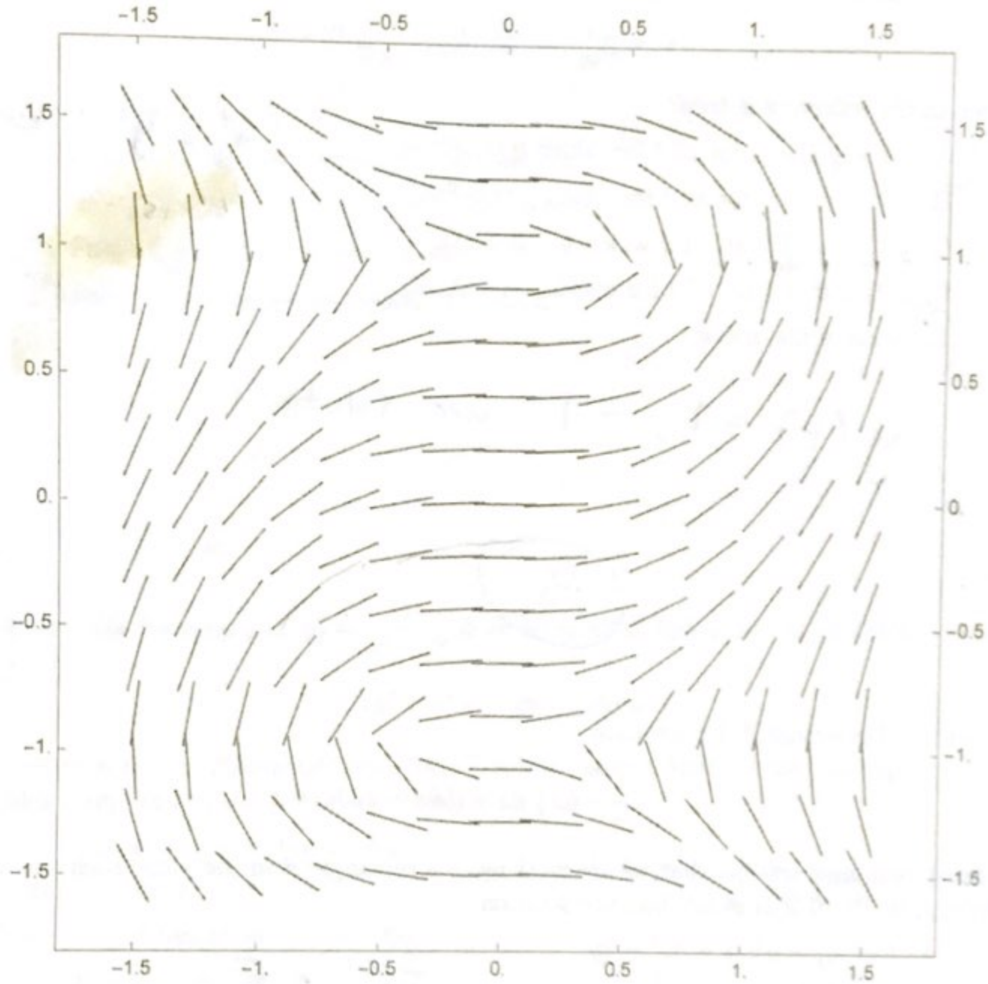
$$= e^{x + \ln x}$$

$$= e^x \cdot x$$

A

3. (4 points) The slope field below corresponds to which differential equation?

$t=1,\ y=1$



~~A.~~ $y' = t^2$

~~B.~~ $y' = ty$

~~C.~~ $y' = \dfrac{y^2}{2t+1}$

~~D.~~ $y' = 4y$

E. $y' = \dfrac{t^2}{1-y^2}$

$t=0 \quad y'=0$

$t=1 \quad y=1$

at $t=0 \to y'=0$

as $t \to \infty \quad y' \to \infty$

E

$-1.5,\ -1.5$

$\dfrac{\frac{9}{4}}{1-\frac{9}{4}} = \dfrac{\frac{9}{4}}{-\frac{5}{4}} = -\dfrac{9}{5}$

$\dfrac{\frac{9}{4}}{1}$

$\dfrac{\frac{9}{4}}{\frac{6}{4}+1}$

4. (3 points) Consider the initial value problem

$$y' = (y^2 - 1)\sin^2(ty), \qquad y(0) = 2.$$

Which of the following is true?

A. $-1 < y(t) < 1$ for all t for which y is defined.

B. $y(t) < \sin^2(t)$ for all t for which y is defined.

C. $y(t) < -1$ for all t for which y is defined.

D. $y(t) > t^2 - 1$ for all t for which y is defined.

E. None of the above.

$y' = y^2 - 1 \sin^2(ty)$

y' is

$y(t) = 1, -1$ are solutions

A

5. (4 points) Determine if the equation

$$\left(\frac{y}{x} + 6x\right) dx + (\ln x - 2)\, dy = 0 \tag{3}$$

is exact (you may assume that we are working in a rectangle R in the plane such that $x > 0$ for all (x, y) in R). If it is exact, find the solution.

A. $F(x, y) = y \ln x + 3x^2 - 2y$

B. $-\frac{y}{x^2} + \frac{1}{x} + 6 = C$

C. $y \ln x + 3x^2 - 2y = C$

D. $F(x, y) = -\frac{y}{x^2} + \frac{1}{x} + 6$

E. The equation is not exact.

$\frac{\partial P}{\partial y} = \frac{1}{x}$

$\frac{\partial Q}{\partial x} = \frac{1}{x}$

$F = \int \ln x - 2 \, dy = y \ln x - 2y + g(x)$

$\frac{\partial F}{\partial x} = \frac{y}{x} + g'(x) = \frac{y}{x} + 6x$

$g'(x) = 6x$

$g(x) = 3x^2$

C

6. (3 points) True or false: there exists a differential equation of the form $y' = f(t, y)$ such that f has continuous partial derivatives on a rectangle R containing $(0, 0)$ and such that

$$y_1 = 2t \quad \text{and} \quad y_2 = 3t \tag{4}$$

are both solutions in R.

A. True. The existence theorem guarantees it.

B. True. The uniqueness theorem guarantees it.

C. False. The existence theorem forbids it.

D. False. The uniqueness theorem forbids it.

7. (4 points) The function $\mu(x, y) = \dfrac{1}{x^2 + y^2}$ is an integrating factor for the equation

$$(x^2 + y^2 - x)dx - y\,dy = 0. \tag{5}$$

Use this to solve the differential equation. You may assume that we are working in a rectangle R which does not contain the point $(0, 0)$.

A. $F(x, y) = x - \arctan(x^2 + y^2)$

B. $x - \frac{1}{2}\ln(x^2 + y^2) = C$

C. $x - \arctan(x^2 + y^2) = C$

D. $F(x, y) = x - \frac{1}{2}\ln(x^2 + y^2)$

E. None of the above

$$\left(1 - \frac{x}{x^2+y^2}\right)dx - \left(\frac{y}{y^2+x^2}\right)dy = 0$$

$$\cancel{\frac{\partial F}{\partial x}} \quad F = \int 1 - \frac{x}{x^2+y^2}\,dx$$

$$\cancel{= x - \int \frac{1}{u}}$$

$$= x - \frac{1}{2}\int \frac{1}{u}\,du$$

$$= x - \frac{1}{2}\ln(x^2+y^2) + g(y)$$

$$\frac{\partial F}{\partial y} = -y\left(\frac{1}{x^2+y^2}\right) + g'(y)$$

$$g'(y) = 0$$

B

Part II: Free Response. Write up a full solution for each problem. A correct answer with an incomplete or incorrect solution will not receive full credit.

8. (6 points) Find the general solution of the linear equation

$$y' - 2y = t^2 e^{2t}.$$

Box your answer

$\cancel{y' - 2y = t^2 e^{2t}}$

$\cancel{\text{Int. factor: } e^{\int -2y} = e^{-y^2}}$

$\cancel{y' - 2ye^{-y^2} = t^2 e^{2t} e^{-y^2}}$

$\cancel{(e^{-y^2} y)' = t^2 e^{2t-y^2}}$

$\cancel{e^{-y^2} y = \int t^2 e^{2t-y^2}\, dt}$

$\cancel{e^{-y^2} y = \frac{t^2}{2} e^{2t-y^2} - \int e^{2t-y^2} t}$

$\cancel{u = e^{2t-y^2} \quad dv = t^2}$

$\cancel{du = 2e^{2t-y^2} \quad v = \frac{1}{3}t^3}$

$\cancel{u = t^2 \quad dv = e^{2t-y^2}}$

$\cancel{du = 2t \quad v = \frac{1}{2}e^{2t-y^2}}$

$\cancel{y' = 2y}$

$\cancel{\frac{dy}{y} = 2t}$

$\cancel{\ln y = 2t}$

$\cancel{y = Ae^{2t}}$

$\cancel{y = v(t)e^{2t}}$

$\cancel{v(t)' e^{2t} + v(t) \cdot 2e^{2t} = 2v(t)e^{2t} + t^2 e^{2t}}$

Variation of params:

hom: $y' = 2y \rightarrow \ln y = 2t$

$y = Ae^{2t}$

$= v(t)e^{2t}$

$v'(t)e^{2t} + v(t)2e^{2t} = 2v(t)e^{2t} + t^2 e^{2t}$

$v'(t) = t^2$

$v(t) = \frac{1}{3}t^3 + C$

$y(t) = \left(\frac{1}{3}t^3 + C\right)e^{2t} = \boxed{\frac{1}{3}t^3 e^{2t} + Ce^{2t}}$

9. (5 points) The differential equation

$$y'' + 2t(y')^2 = 0$$

is an example of a *second order differential equation*. The change of variable $v = y'$ (so that $v' = y''$) turns this equation into a first-order separable equation. Using this change of variable, find the particular solution which satisfies

$$y(1) = \frac{\pi}{2}, \qquad y'(0) = 1.$$

Box your answer

$v = y' \rightarrow \quad v' + 2t(v^2) = 0$

Separable:

$v' = -2tv^2$

$\frac{dv}{dt} = -2tv^2$

$\frac{dv}{v^2} = -2t\,dt$

$-\frac{1}{v} = -t^2 + C$

$-\frac{1}{-t^2+C} = v(t)$

$v(t) = -\frac{1}{C-t^2}$

$y' = -\frac{1}{C-t^2}$

$y'(0) = 1 \rightarrow \quad 1 = -\frac{1}{C}$

$C = -1$

$y'(t) = -\frac{1}{-t^2-1} = \frac{1}{t^2+1}$

$y(t) = \int \frac{1}{t^2+1} = \arctan(t) + C$

$y(1) = \frac{\pi}{2}$

$\arctan(1) + C = \frac{\pi}{2}$

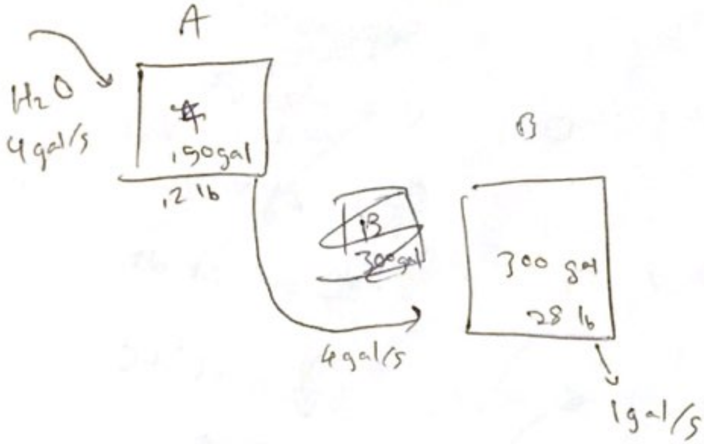
$\frac{\pi}{4} + C = \frac{\pi}{2} \rightarrow C = \frac{\pi}{4}$

$\therefore \boxed{y(t) = \arctan(t) + \frac{\pi}{4}}$

5

10. (8 points) Consider two tanks, labeled tank A and tank B. Tank A contains 150 gal of solution in which is dissolved 12 lbs of salt. Tank B contains 300 gal of solution in which is dissolved 28 lbs of salt. Pure water flows into tank A at a rate of 4 gal/s. There is a drain at the bottom of tank A. The solution leaves tank A via this drain at a rate of 4 gal/s and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B at a rate of 1 gal/s. **Set up, but do not evalute,** a system of differential equations involving the variables x (amount of salt in tank A), y (amount of salt in tank B), and t (time in seconds). Do not forget to include any initial conditions. Box your entire answer

8



A: rate in = 0

rate out = $4\,\text{gal/s} \times \dfrac{x(t)}{150\,\text{gal}}$

$$\frac{dx}{dt} = -\frac{4}{150}x(t)$$

B: rate in = $-\dfrac{4}{150}x(t)$

rate out = ~~1 gal/s × 300~~

$1\,\frac{\text{gal}}{\text{s}} \times \dfrac{y(t)}{300+3t}$

$$\frac{dy}{dt} = -\frac{4}{150}x(t) - \frac{y(t)}{300+3t}$$

$V(t) = 300+3t$

since 4-1 diff in rates

$$\frac{dx}{dt} = \frac{4}{150}x(t) \to \cancel{x(t)}$$

$$\ln x = \frac{4}{150}t$$

$$x = Ae^{\frac{4}{150}t}$$

$$x(0) = \cancel{35}\ 12$$

$$x = 12e^{\frac{4}{150}t}$$

$$\frac{dx}{dt} = \frac{4}{150}\cdot 12e^{\frac{4}{150}t}$$

$$\frac{dy}{dt} = \frac{4}{150}\cdot 12e^{\frac{4}{150}t} - \frac{y(t)}{300+3t}$$

$x(t) = x$

$y(t) = y$

$$\boxed{\begin{cases} \dfrac{dx}{dt} = -\dfrac{4}{150}x(t) \\[2ex] \dfrac{dy}{dt} = -\dfrac{4}{150}x(t) - \dfrac{y(t)}{300+3t} \end{cases}}$$

initial conditions:

$x(0) = 12$

$y(0) = 28$

11. (6 points) The differential equation

$$yy' = -y^2 + 2e^{-t} \tag{6}$$

is an example of a Bernoulli equation with degree $n = -1$.

(a) The change of variables $z = y^2$ turns this equation into a first-order linear differential equation. Using this, write the linear equation above in the form $z' = a(t)z + f(t)$. (I will not be grading your work for this problem, just your answer). Box your answer.

$z = y^2$ $\quad \cancel{\frac{dz}{dt}} = z' = 2yy'$ $\quad \frac{1}{2}z' = yy'$

$\frac{1}{2}z' = -z + 2e^{-t}$

$\cancel{z' = 2z + 4e^{-t}}$

$$\boxed{z' = -2z + 4e^{-t}}$$

(b) Solve the differential equation. Your answer should be in the form $\boxed{y(t) = \cdots}$

$z' = -2z + 4e^{-t}$ $\qquad z' = -2z$

$v'(t)e^{-2t} + \cancel{v(t)(-2e^{-2t})} = \cancel{-2v(t)e^{-2t}} + 4e^{-t}$ $\qquad \int \frac{dz}{z} = \int -2\,dt$

$\ln z = -2t$

$z = Ae^{-2t}$

$= v(t)e^{-2t}$

$v'(t)e^{-2t} = 4e^{-t}$

$v'(t) = \frac{4}{e^t} \times e^{2t}$

$v'(t) = \cancel{4e^{2}}\ 4e^t$

$v(t) = \int 4e^t = 4e^t + C$ $\qquad y^2 = 4e^{-t} + Ce^{-2t}$

$z = 4e^t e^{-2t} + Ce^{-2t}$ $\qquad \cancel{y(t)\ \sqrt{4e^{-t} + Ce^{-2t}}}$

$z = 4e^{-t} + Ce^{-2t}$

$$\boxed{y(t) = \sqrt{4e^{-t} + Ce^{-2t}}}$$