MATH 33A Midterm 2 Feb. 25, 2019

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DISCUSSION SECTION NUMBER: 3F (Fan Zhou)

Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

Page	Points	Score
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4	4	4
5	8	8
6	6	4
7	10	8
8	6	6
Total:	50	44

1. Consider the matrix

$$A = \begin{pmatrix} 1 & 5 & -3 \\ 2 & 0 & 4 \\ 2 & 2 & 2 \end{pmatrix}.$$

4 (a) [4 pts] Find a basis for $\ker(A)$, which is one-dimensional.

$A\vec{x} = \vec{0} \rightarrow RREF\ (A \,\vdots\, \vec{0})$

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & 0 \\ 2 & 0 & 4 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right) \xrightarrow[-2\cdot R1]{-2\cdot R1} \left(\begin{array}{ccc|c} 1 & 5 & -3 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & -8 & 8 & 0 \end{array}\right) \xrightarrow{\div -10} \left(\begin{array}{ccc|c} 1 & 5 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -8 & 8 & 0 \end{array}\right) \xrightarrow[+8\cdot R2]{-5\cdot R2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\rightarrow$ $x_1 + 2t = 0$, $x_2 - t = 0$, $x_3 = t$ $\Rightarrow \vec{x} = \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$\boxed{\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}}$ is a basis for $\ker(A)$ ✓

0 (b) [2 pts] Using your kernel computation, or otherwise, write down a linear relation between the columns of A that shows that the third column can be viewed as redundant. No further explanation necessary.

$$\begin{pmatrix} 1 & 5 & -3 \\ 2 & 0 & 4 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + 1\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = \vec{0}$$

should write third column = ______

$$\boxed{-2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + 1\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = \vec{0}}$$

$(-2 \neq 0,\ 1 \neq 0,\ 1 \neq 0)$

2 (c) [2 pts] Write down a basis for $\mathrm{im}(A)$. You may use any of the previous parts of the problem with no further explanation, even if you couldn't solve them. Or you may solve this problem from scratch using row reduction.

$$\mathrm{im}(A) = \mathrm{Span}\left(\left\{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}\right\}\right) = \mathrm{Span}\left(\left\{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}\right\}\right)$$

(3rd column is only redundant column)

✓ $\boxed{\left\{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}\right\}}$

$\left(\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} \neq k\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \text{ for } k \in \mathbb{R}\right)$

(d) [8 pts] Convert your answer from the previous part of the problem into an orthonormal basis for im(A). (Keep your work well-organized!)

$$\vec{v}_1 = \begin{pmatrix}1\\2\\2\end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix}5\\0\\2\end{pmatrix}$$

$$\|\vec{v}_1\| = \sqrt{1+4+4} = \sqrt{9} = 3 \qquad \vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}1/3\\2/3\\2/3\end{pmatrix}$$

$$\vec{u}_1 \cdot \vec{v}_2 = \begin{pmatrix}5\\0\\2\end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix} = \frac{1}{3}(5+0+4) = \frac{1}{3}(9) = 3$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{pmatrix}5\\0\\2\end{pmatrix} - (3)\left(\frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}\right) = \begin{pmatrix}5\\0\\2\end{pmatrix} - \begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}4\\-2\\0\end{pmatrix}$$

$$\|\vec{v}_2^{\perp}\| = \sqrt{16+4+0} = \sqrt{20} = 2\sqrt{5}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{1}{2\sqrt{5}}\begin{pmatrix}4\\-2\\0\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\-1\\0\end{pmatrix} = \begin{pmatrix}2/\sqrt{5}\\-1/\sqrt{5}\\0\end{pmatrix}$$

$$\boxed{\left\{ \begin{pmatrix}1/3\\2/3\\2/3\end{pmatrix}, \begin{pmatrix}2/\sqrt{5}\\-1/\sqrt{5}\\0\end{pmatrix} \right\}}$$

(e) [4 pts] Use your work from the previous part of the problem to write a $M = QR$ factorization for a relevant matrix M. (Part of the problem is deciding what M should be!)

Let $M = \begin{pmatrix} 1 & 5 \\ 2 & 0 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix}$.

Then $M = QR$, where $Q = \begin{pmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{pmatrix}$ and $R = \begin{pmatrix} \|\vec{v}_1\| & \vec{u}_1 \cdot \vec{v}_2 \\ 0 & \|\vec{v}_2^{\perp}\| \end{pmatrix}$. Thus

$$\boxed{\begin{pmatrix} 1 & 5 \\ 2 & 0 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1/3 & 2/\sqrt{5} \\ 2/3 & -1/\sqrt{5} \\ 2/3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 0 & 2\sqrt{5} \end{pmatrix}}$$

We can check that this works by just doing the matrix multiplication.

2. Suppose we have a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, and suppose further that we have a subspace V in the range $\mathbb{R}^n$. We will define a new subset of the *domain* $\mathbb{R}^m$ called the *pre-image* of V as follows:

$$\mathrm{PreIm}(V) := \{\vec{x} \text{ in } \mathbb{R}^m \mid T(\vec{x}) \text{ is in } V\}.$$

That is, $\mathrm{PreIm}(V)$ consists of all vectors in $\mathbb{R}^m$ that get mapped by T to a vector that is in V.

(a) [8 pts] Prove that $\mathrm{PreIm}(V)$ is a subspace of the domain $\mathbb{R}^m$ by checking all three necessary conditions.

- Since V is a subspace, $\vec{0} \in V$. Since T is a linear transformation, $T(\vec{0}) = \vec{0}$. Thus $T(\vec{0}) \in V$, and thus $\vec{0} \in \mathrm{PreIm}(V)$.
- Let $\vec{x}, \vec{y}$ be any two vectors in $\mathrm{PreIm}(V)$. Then $T(\vec{x}) \in V$ and $T(\vec{y}) \in V$, so since V is a subspace, $(T(\vec{x}) + T(\vec{y})) \in V$. But T is a linear transformation, so $T(\vec{x}) + T(\vec{y}) = T(\vec{x} + \vec{y})$ and thus $T(\vec{x} + \vec{y}) \in V$. Thus $(\vec{x} + \vec{y}) \in \mathrm{PreIm}(V)$
- Let $\vec{x}$ be any vector in $\mathrm{PreIm}(V)$, and let k be any constant $k \in \mathbb{R}$. Then $T(\vec{x}) \in V$, so since V is a subspace, $(k\,T(\vec{x})) \in V$. But T is a linear transformation, so $k\,T(\vec{x}) = T(k\vec{x})$ and thus $T(k\vec{x}) \in V$. Thus $k\vec{x} \in \mathrm{PreIm}(V)$.

Since $\mathrm{PreIm}(V)$ contains $\vec{0}$ and is closed under addition and scalar multiplication, it is a subspace of $\mathbb{R}^m$.

2 (b) [2 pts] If we had chosen the subspace V to be all of the range $\mathbb{R}^n$, what subspace would $\mathrm{PreIm}(V)$ be? No explanation necessary.

$\boxed{\text{all of } \mathbb{R}^m}$ ✓

2 (c) [2 pts] If we had chosen the subspace V to be the zero subspace $\{\vec{0}\}$ in the range $\mathbb{R}^n$, what subspace would $\mathrm{PreIm}(V)$ be? No explanation necessary.

$\boxed{\ker(T)}$ ✓

0 (d) [2 pts] Circle the one correct statement out of the four choices below (hint: think about $T : \mathrm{PreIm}(V) \to V$ rather than $T : \mathbb{R}^m \to \mathbb{R}^n$).

- $\dim(\mathrm{PreIm}(V)) = \dim(V)$ regardless of V and/or T.
- X $\dim(\mathrm{PreIm}(V)) \geq \dim(V)$ regardless of V and/or T. T does not need to surject onto V
- $\dim(\mathrm{PreIm}(V)) \leq \dim(V)$ regardless of V and/or T.
- In some cases $\dim(\mathrm{PreIm}(V)) > \dim(V)$. In other cases $\dim(\mathrm{PreIm}(V)) < \dim(V)$. Finally, there are also cases where $\dim(\mathrm{PreIm}(V)) = \dim(V)$.

3. Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be two linear transformations that treat this basis in the following manner:

$$T(\vec{v}_1) = \vec{v}_2, \quad T(\vec{v}_2) = \vec{v}_1, \qquad S(\vec{v}_1) = \vec{v}_1 - \vec{v}_2, \quad S(\vec{v}_2) = \vec{v}_2$$

2 (a) [4 pts] Write down the matrix B for the composition $T \circ S$ in $\mathcal{B}$-coordinates.

$$(T(\vec{v}_1))_{\mathcal{B}} = (\vec{v}_2)_{\mathcal{B}} = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)_{\mathcal{B}} \qquad (T(\vec{v}_2))_{\mathcal{B}} = (\vec{v}_1)_{\mathcal{B}} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)_{\mathcal{B}}$$

$$(S(\vec{v}_1))_{\mathcal{B}} = (\vec{v}_1 - \vec{v}_2)_{\mathcal{B}} = \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)_{\mathcal{B}} \qquad S(\vec{v}_2))_{\mathcal{B}} = (\vec{v}_2)_{\mathcal{B}} = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)_{\mathcal{B}}$$

Let B_T be the matrix for T in $\mathcal{B}$-coordinates, and let B_S be the matrix for S in $\mathcal{B}$-coordinates. Then

$$B_T = \begin{pmatrix} | & | \\ T(\vec{v}_1)_{\mathcal{B}} & T(\vec{v}_2)_{\mathcal{B}} \\ | & | \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad B_S = \begin{pmatrix} | & | \\ S(\vec{v}_1)_{\mathcal{B}} & S(\vec{v}_2)_{\mathcal{B}} \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

The matrix B for the composition $T \circ S$ in $\mathcal{B}$ coordinates is just $B_S B_T$, so

$$B = B_S B_T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0+0 & 1+0 \\ 0+1 & -1+0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}}.$$

wrong order

-2

6

(b) [6 pts] Prove that, if we assume that S was an orthogonal transformation, then the basis vectors $\vec{v}_1$ and $\vec{v}_2$ *cannot* be orthogonal to each other.

Since S is orthogonal, it preserves lengths. Therefore

$$\|\vec{v}_1\|^2 = \|S(\vec{v}_1)\|^2 = \|\vec{v}_1 - \vec{v}_2\|^2 = (\vec{v}_1 - \vec{v}_2)\cdot(\vec{v}_1 - \vec{v}_2)$$

$$= \vec{v}_1 \cdot \vec{v}_1 - \vec{v}_2 \cdot \vec{v}_1 - \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_2 \cdot \vec{v}_2 = \|\vec{v}_1\|^2 + \|\vec{v}_2\|^2 - 2\,\vec{v}_1 \cdot \vec{v}_2$$

Rearranging, we have

$$\|\vec{v}_1\|^2 = \|\vec{v}_1\|^2 + \|\vec{v}_2\|^2 - 2\,\vec{v}_1 \cdot \vec{v}_2$$

$$\vec{v}_1 \cdot \vec{v}_2 = \tfrac{1}{2}\|\vec{v}_2\|^2$$

Since $\vec{v}_2$ is a basis vector, it cannot have length 0 (since otherwise $\vec{v}_2$ would be redundant and $\{\vec{v}_1, \vec{v}_2\}$ would not be a basis). But then

$$\vec{v}_1 \cdot \vec{v}_2 \neq 0,$$

so $\vec{v}_1$ and $\vec{v}_2$ cannot be orthogonal to each other.

4. Multiple choice and/or true and false (circle one answer only; no justification needed). In all of the problems below, A is an nxm matrix.

   (a) [2 pts] What can we say about $\dim(\text{im}(A)) + \dim(\ker(A))$?

   Always $= n$ Always $= m$ Neither of these

   (b) [2 pts] What can we say about $\dim(\ker(A))$?

   Always $< n$ Always $> n$ Neither of these

   (c) [2 pts] If B is a pxn matrix, then we must have $\text{rank}(BA) \leq \text{rank}(A)$.

   TRUE FALSE