MATH 33A Midterm 2 Nov. 20, 2017

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DISCUSSION SECTION NUMBER: 2D

Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

Page	Points	Score
2	4	4
3	8	8
4	6	4
5	16	11
6	10	10
7	6	6
8	10	6
Total:	60	49

1. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 4 \end{pmatrix}$$

(a) [4 pts] Find a basis for Im(A).

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 4 \end{pmatrix} \xrightarrow[-(II)]{-(I)} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xrightarrow[-(III)]{\div(2)}$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ correspond to $\underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}}_{\text{basis}}$ in A

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(b) [8 pts] Find an orthonormal basis for Im(A).

$$\vec{v_1} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix} \qquad \vec{u_1} = \frac{1}{\|\vec{v_1}\|}\vec{v_1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}.$$

$$\vec{v_2}^{\perp} = \vec{v_2} - (v_2 \cdot \vec{u_1})\,\vec{u_1}$$

$$= \begin{pmatrix}0\\1\\0\\1\end{pmatrix} - \begin{pmatrix}0\\1\\0\\1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\0\\1\end{pmatrix} - 0$$

$$= \begin{pmatrix}0\\1\\0\\1\end{pmatrix} \qquad \vec{u_2} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

$$v_3^{\perp} = \vec{v_3} - (\vec{v_3} \cdot \vec{u_1})\,\vec{u_1} - (\vec{v_3} \cdot \vec{u_2})(\vec{u_2})$$

$\frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$

$$= \begin{pmatrix}1\\2\\3\\4\end{pmatrix} - \begin{pmatrix}1\\2\\3\\4\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{pmatrix} - \begin{pmatrix}1\\2\\3\\4\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\begin{pmatrix}0\\1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$$

$\frac{6}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$

$$= \begin{pmatrix}1\\2\\3\\4\end{pmatrix} - \left(\frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}}\right)\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{pmatrix} - \left(\frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}}\right)\begin{pmatrix}0\\1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\3\\4\end{pmatrix} - \begin{pmatrix}2\\0\\2\\0\end{pmatrix} - \begin{pmatrix}0\\3\\0\\3\end{pmatrix}$$

$$= \begin{pmatrix}-1\\-1\\1\\1\end{pmatrix} \qquad \vec{u_3} = \frac{1}{2}\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}$$

(c) [6 pts] Find the orthogonal projection of the vector $\vec{x}$ onto Im(A), for $\vec{x}$ given below:

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$ This is $\vec{x}^{\perp}$

$$\text{proj}_{\text{Im}(A)}(\vec{x}) = \cancel{\vec{x}} - (\vec{u_1}\cdot\vec{x})\vec{u_1} - (\vec{u_2}\cdot\vec{x})\vec{u_2} - (\vec{u_3}\cdot\vec{x})\vec{u_3}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$- \left(\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$- \left(\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} - 0\cdot\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

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2. Suppose V is a subspace of $\mathbb{R}^n$, with $\dim(V) = m$. Let $V^\perp$ denote the orthogonal complement of V. Let P_V and $P_{V^\perp}$ denote the matrices corresponding to orthogonal projections onto V and $V^\perp$, respectively.

(a) [8 pts] Determine the following items (no work necessary):

$\mathrm{Im}(P_V) =$ V $\mathrm{rank}(P_V) =$ m

$\ker(P_V) =$ $V^\perp$ $\mathrm{nullity}(P_V) =$ n−m

$\mathrm{Im}(P_{V^\perp}) =$ ~~$PV^\perp$~~ $\mathrm{rank}(P_{V^\perp}) =$ m

$\ker(P_{V^\perp}) =$ ~~PV~~ $\mathrm{nullity}(P_{V^\perp}) =$ n−m

−4

(b) [4 pts] What matrix would be the result of the matrix multiplication $(P_{V^\perp})(P_V)$? Explain in terms of images and kernels. (Hint: Consider $(P_{V^\perp})(P_V)(\vec{x})$ for an arbitrary vector $\vec{x}$ in $\mathbb{R}^n$)

$(P_{V^\perp})\underbrace{(P_V)(\vec{x})}_{\text{Im is } V} = \vec{0}$

V is $\perp$ to $V^\perp$ so $P_{V^\perp}(V) = \vec{0}$

$(P_{V^\perp})(P_V) = \vec{0}$ OK

(c) [4 pts] What matrix would be the result of the matrix multiplication $(P_V)(P_V)$? Explain.

$(P_V)(P_V)\vec{x} = P_V(\vec{x})$

$(P_V)\vec{x}$ is on V.

$P_V[P_V(\vec{x})] = P_V(\vec{x})$

−1

3. Let $\mathfrak{B}$ be the basis of $\mathbb{R}^2$ given below

$$\mathfrak{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \end{pmatrix} \right\} \quad \left[\begin{matrix} 1 & 1 \\ 0 & 1/3 \end{matrix}\right) = \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right]$$

(a) [6 pts] Suppose A (given below) is the matrix for some linear transformation T in standard coordinates. Find the matrix B for T in $\mathfrak{B}$-coordinates.

$$A = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & -3 \\ 0 & 3 \end{pmatrix}$$

$$B = S^{-1}AS \qquad S^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 0 & 3 \end{bmatrix} \qquad = \begin{bmatrix} 1 & 1 \\ 0 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 \\ 1/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 9 \\ 1/3 & 0 \end{bmatrix}$$

(b) [4 pts] Suppose T' is a different linear transformation that swaps our two basis vectors. That is, we have $T'\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ and $T'\left(\begin{pmatrix} -3 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Find the matrix B' for T' in $\mathfrak{B}$-coordinates.

$$T(\vec{v_1}) = \vec{v_2} \rightarrow [T(\vec{v_1})]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(\vec{v_2}) = \vec{v_1} \rightarrow [T(\vec{v_2})]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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4. [6 pts] Consider the following table of data for heights, gender, and average body temperatures of some young adults:

Height h (inches above 5')	Gender g (1 for females, 0 for males)	Avg body temp t (degrees F away from 98)
2	1	-1
2	0	2
1	1	0
3	0	1

Suppose you would like to fit this data to a function of the form

$$t = c_0 + c_1 h + c_2 g$$

using least squares. Set up (but do not solve) the normal equation that you need to solve to find c_0^*, c_1^*, c_2^*. Your answer should be in the form:

$$(M)(\vec{c}^*) = (\vec{v})$$

where you have filled in M and $\vec{v}$, but did not solve for $\vec{c}^*$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 3 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M = A^T A = \underset{3\times 4}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 3 \\ 1 & 0 & 1 & 0 \end{bmatrix}} \underset{4\times 3}{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix}} = \begin{bmatrix} 4 & 8 & 2 \\ 8 & 18 & 3 \\ 2 & 3 & 2 \end{bmatrix}$$

$$\vec{v} = A^T \vec{b} = \underset{3\times 4}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 3 \\ 1 & 0 & 1 & 0 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$$

5. [4 pts] Suppose we have three bases for $\mathbb{R}^n$: the standard basis $\mathcal{E}$, and two others $\mathfrak{B}$ and $\mathfrak{C}$. Suppose that the change of basis matrices are $S : \mathfrak{B} \to \mathcal{E}$ and $T : \mathfrak{C} \to \mathcal{E}$ respectively. That is to say, for any vector $\vec{x}$ in $\mathbb{R}^n$,

$$S(\vec{x})_{\mathfrak{B}} = (\vec{x})_{\mathcal{E}}, \qquad T(\vec{x})_{\mathfrak{C}} = (\vec{x})_{\mathcal{E}}$$

What matrix changes basis from $\mathfrak{B}$ to $\mathfrak{C}$? In other words, find the matrix M (in terms of S and T) that would make

$$M(\vec{x})_{\mathfrak{B}} = (\vec{x})_{\mathfrak{C}}$$

for all vectors $\vec{x}$.

Hint: A triangular diagram similar to one from class might be helpful.

$S((\vec{x})_B) = T(\vec{x})_C$

$(\vec{x})_C = T^{-1}T(\vec{x})_C = T^{-1}S(\vec{x})_B$

0

$M = T^{-1}S$

6. True or false (circle your answer; no justification needed). In all of the problems below, A is an nxn *square* matrix.

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(a) [2 pts] If the matrix A is orthogonal, then $A\vec{v} \cdot A\vec{w} = \vec{v} \cdot \vec{w}$ for any two vectors $\vec{v}, \vec{w}$ in $\mathbb{R}^n$.

TRUE FALSE

(b) [2 pts] If A is an orthogonal matrix, then both A and A^T are invertible.

TRUE FALSE

(c) [2 pts] If A and A^T are both invertible matrices, then A is orthogonal.

TRUE FALSE $\begin{pmatrix} 1 & -3 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

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