MATH 33A Midterm 1 Jan. 28, 2019

STUDENT NAME: ________________

STUDENT ID NUMBER: ________________

DISCUSSION SECTION NUMBER: 3F

Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

Page	Points	Score
2	6	6
3	5	5
4	13	13
5	8	8
6	8	8
7	10	10
Total:	50	50

1. [6 pts] Is the vector $\vec{\mathbf{b}} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$ a linear combination of the vectors $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{w}} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$? If so, write down the linear combination in the format $\vec{\mathbf{b}} = c_1\vec{\mathbf{v}} + c_2\vec{\mathbf{w}}$. If not, explain why not.

Writing $\vec{b}$ as a combination of $\vec{v}$ and $\vec{w}$ is equivalent to finding a vector $\vec{x} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ such that $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} | & | \\ \vec{v} & \vec{w} \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

We attempt to do this by RREF'ing the augmented matrix $\left(A \,\vdots\, \vec{b}\right)$:

$$\left(\begin{array}{cc:c} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 1 & 1 & 2 \end{array}\right) \xrightarrow[-R1]{} \left(\begin{array}{cc:c} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{array}\right) \xrightarrow[+2\cdot R2]{-3\cdot R2} \left(\begin{array}{cc:c} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow c_1 = 5,\ c_2 = -3 \Rightarrow \vec{x} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

We can confirm that

$$\boxed{\vec{b} = 5\vec{v} + (-3)\vec{w}}$$

since $5\vec{v} - 3\vec{w} = 5\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 9 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix} = \vec{b}$, ✓

So we're done.

2. [5 pts] Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

Circle all of the following vectors which are members of ker(A).

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{z} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$A\vec{v} = \begin{pmatrix} 1+2-1+1 \\ -1+0+1+1 \\ 0+0+0+0 \\ 3+4-3+1 \\ 1+1-1+0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad A\vec{w} = \vec{0}$$

$$A\vec{x} = \begin{pmatrix} 2+0-2+0 \\ -2+0+2+0 \\ 0+0+0+0 \\ 6+0-6+0 \\ 2+0-2+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad A\vec{y} = \begin{pmatrix} 1+0-2+1 \\ -1+0+2+1 \\ 0+0+0+0 \\ 3+0-6+1 \\ 1+0-2+0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A\vec{z} = \begin{pmatrix} 1-2+0+1 \\ -1+0+0+1 \\ 0+0+0+0 \\ 3-4+0+1 \\ 1-1+0+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

3. Suppose you know that $A^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & 2 \end{pmatrix}$.

6 (a) [6 pts] Find $(AB)^{-1}$.

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2+1 & -1+1 \\ 6+2 & 3+2 \end{pmatrix} = \boxed{\begin{pmatrix} -1 & 0 \\ 8 & 5 \end{pmatrix}}$$

5 (b) [5 pts] Find B.

$B = (B^{-1})^{-1}$, so we RREF the matrix $\left(B^{-1} \,\vdots\, I_n\right)$:

$$\left(\begin{array}{cc:cc} -1 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right) \xrightarrow{\div -1} \left(\begin{array}{cc:cc} 1 & -1 & -1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right) \xrightarrow[-3 \cdot R1]{} \left(\begin{array}{cc:cc} 1 & -1 & -1 & 0 \\ 0 & 5 & 3 & 1 \end{array}\right) \div 5$$

$$\left(\begin{array}{cc:cc} 1 & -1 & -1 & 0 \\ 0 & 1 & 3/5 & 1/5 \end{array}\right) \xrightarrow{+R2} \left(\begin{array}{cc:cc} 1 & 0 & -2/5 & 1/5 \\ 0 & 1 & 3/5 & 1/5 \end{array}\right)$$

Since the result should be $\left(I_n \,\vdots\, (B^{-1})^{-1}\right)$, we have

$$\boxed{B = \begin{pmatrix} -2/5 & 1/5 \\ 3/5 & 1/5 \end{pmatrix}}$$

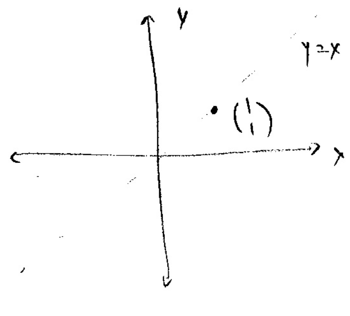
2 (c) [2 pts] What was the rank of A? (this should require no computations)

Since A is invertible and A is a 2×2 matrix, $\text{rank}(A) = \boxed{2}$

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates all vectors about the origin *counter-clockwise* by $\pi/2$. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects all vectors about the line $y = x$.

6

(a) [6 pts] Find the single matrix representing the composite function $T \circ R$. *Hint: A geometric approach may be easier than a computation.*



Projection onto the line defined by $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rightarrow \frac{1}{w_1^2 + w_2^2}\begin{pmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{pmatrix}$

$\Rightarrow$ Matrix $\text{proj}_{y=x} = \frac{1}{1^2+1^2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

Rotation counterclockwise by $\theta \rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\Rightarrow$ Matrix $\text{rot}_{\pi/2} = \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(Matrix for reflection) = 2 (Projection matrix) $- I_2$ (for 2x2 matrices)

$\Rightarrow$ Matrix $\text{refl}_{y=x} = 2\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Since R corresponds to the matrix ($\text{refl}_{y=x}$) and T corresponds to the matrix ($\text{rot}_{\pi/2}$), $T \circ R$ corresponds to the matrix

$$(\text{rot}_{\pi/2})(\text{refl}_{y=x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}$$

2

(b) [2 pts] Find $T(R(\vec{v}))$ where $\vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

$$T(R(\vec{v})) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}$$

Matrix corresponding to $T \circ R$

(c) [6 pts] Find the single matrix representing the composite function $R \circ T$. *Hint: A geometric approach may be easier than a computation.*

Given the work done in part (a), $R \circ T$ corresponds to the matrix

$$(\text{refl}_{y=x})(\text{rot}_{\pi/2}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

(since T corresponds to the matrix $(\text{rot}_{\pi/2})$ and R corresponds to the matrix $(\text{refl}_{y=x})$).

(d) [2 pts] Find $R(T(\vec{v}))$ where $\vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

$$R(T(\vec{v})) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \boxed{\begin{pmatrix} -2 \\ -3 \end{pmatrix}}$$

Matrix corresponding to $R \circ T$

5. [4 pts] Suppose you know that $\vec{w}$ is in $\ker(B)$, and you also know that $B\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Use this information to find $B(2\vec{w} - 3\vec{v})$.

$$B(2\vec{w} - 3\vec{v}) = B(2\vec{w}) - B(3\vec{v})$$
$$= 2(B\vec{w}) - 3(B\vec{v})$$
$$= 2(\vec{0}) - 3(B\vec{v}) \qquad (\vec{w} \in \mathrm{Ker}(B) \Rightarrow B\vec{w} = \vec{0})$$
$$= -3\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$
$$= \boxed{\begin{pmatrix} -3 \\ -6 \\ -9 \end{pmatrix}} \checkmark$$

6. True or false (circle your answer; no justification needed). In all of the problems below, A is an nxn *square* matrix.

(a) [2 pts] If A is the coefficient matrix for some linear system, and $\mathrm{rank}(A) = n$, then the system has a unique solution.

TRUE FALSE

(b) [2 pts] If A is the coefficient matrix for some linear system, and $\mathrm{rank}(A) < n$, then the system must have infinitely many solutions.

TRUE FALSE

(c) [2 pts] If A is the coefficient matrix for some linear system, and the system had a unique solution, then the RREF of A must be precisely the identity matrix $I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$.

TRUE FALSE