MATH 33A Midterm 1 Jan. 29, 2018

STUDENT NAME: SOLUTIONS

STUDENT ID NUMBER: ______

DISCUSSION SECTION NUMBER: ______

Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

Page	Points	Score
2	9	
3	7	
4	6	
5	8	
6	10	
7	10	
Total:	50	

1. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation where it is known that $T\left(\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ and $T\left(\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}\right) = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$.

   (a) [6 pts] Build the matrix A for the transformation T.

$$T(\vec{e}_1) = T\left(\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}\right) = T\left(\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}\right) + T\left(\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}\right)$$
$$= \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$

$$T(\vec{e}_2) = T\left(\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} - \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}\right) = T\left(\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}\right) - T\left(\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}\right)$$
$$= \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 5 & 3 \\ 3 & 1 \end{pmatrix}}$$

   (b) [3 pts] Find $T\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)$.

$$T\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right) = A\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 5 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} -3 \\ 7 \\ 5 \end{pmatrix}}$$

2. (a) [5 pts] Write down the three conditions needed for a collection of vectors V in $\mathbb{R}^n$ to be a subspace. Be precise!

1) THE ZERO VECTOR $\vec{0}$ MUST BE IN V

2) IF $\vec{v}_1$ AND $\vec{v}_2$ ARE IN V, THEN $(\vec{v}_1 + \vec{v}_2)$ MUST ALSO BE IN V

3) IF $\vec{v}$ IS IN V AND k IS ANY SCALAR, THEN $k\vec{v}$ MUST ALSO BE IN V.

(b) [2 pts] Suppose that $T : \mathbb{R}^{11} \to \mathbb{R}^{345}$ is a linear function, so that $\text{im}(T)$ is a subspace of $\mathbb{R}^{??}$ and $\ker(T)$ is a subspace of $\mathbb{R}^{??}$. Fill in the question marks. Make sure it is clear in your answer which is which!

IM(T) IS A SUBSPACE OF $\mathbb{R}^{345}$.

KER(T) IS A SUBSPACE OF $\mathbb{R}^{11}$

(c) [6 pts] Prove directly from the definition of a subspace that, in $\mathbb{R}^3$, the xy-plane (ie $V = \left\{ \text{vectors } \vec{v} \text{ of the form } \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} \right\}$) is indeed a subspace.

1) $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ IS IN V USING $V_1 = 0, V_2 = 0$

2) IF $\vec{V}$ IS IN V, $\vec{V} = \begin{pmatrix} V_1 \\ V_2 \\ 0 \end{pmatrix}$. IF $\vec{\omega}$ ~~[illegible]~~ IS IN V, $\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ 0 \end{pmatrix}$.

THEN $\vec{V} + \vec{\omega} = \begin{pmatrix} V_1 + \omega_1 \\ V_2 + \omega_2 \\ 0 + 0 \end{pmatrix} = \begin{pmatrix} V_1 + \omega_1 \\ V_2 + \omega_2 \\ 0 \end{pmatrix}$ IS ALSO IN V SINCE THE THIRD COORDINATE IS STILL 0.

3) IF $\vec{V}$ IS IN V, $\vec{V} = \begin{pmatrix} V_1 \\ V_2 \\ 0 \end{pmatrix}$. IF k IS A SCALAR, $k\vec{V} = \begin{pmatrix} kV_1 \\ kV_2 \\ 0 \end{pmatrix}$ IS ALSO IN V SINCE THE THIRD COORDINATE IS STILL 0.

3. [8 pts] Use Gauss-Jordan row reduction to solve the linear system below.

$$x_1 + 2x_2 + 3x_3 = 1$$
$$4x_1 + 5x_2 + 6x_3 = 1$$
$$7x_1 + 8x_2 + 9x_3 = 1$$

If there are any free variables, name them s and/or t. Write your final answer in vector form $\vec{\mathbf{x}} = \begin{pmatrix} ?? \\ ?? \\ ?? \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{array}\right) \xrightarrow[-7(R1)]{-4(R1)} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & -12 & -6 \end{array}\right) \xrightarrow{\div(-3)} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -6 & -12 & -6 \end{array}\right) \xrightarrow[+6(R2)]{-2(R2)}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

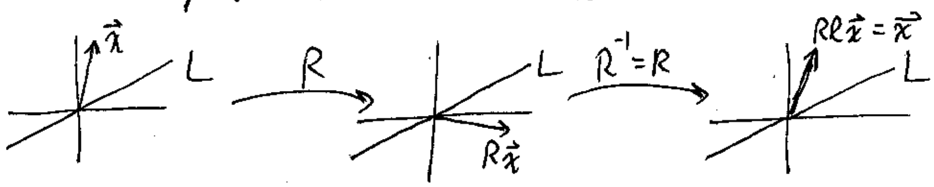
FREE, LET $x_3 = t$. THEN $x_1 = -1 + t$, $x_2 = 1 - 2t$

SO $\boxed{\vec{x} = \begin{pmatrix} -1+t \\ 1-2t \\ t \end{pmatrix}}$

4. Consider the xy-plane $\mathbb{R}^2$. Suppose L is some line through the origin spanned by the vector $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. Suppose further that $\vec{w}$ is a unit vector (so $w_1^2 + w_2^2 = 1$).

(a) [4 pts] If R is the matrix for reflection about L, explain using geometric intuition (rather than algebraic formulas) what R^{-1} should be.

R^{-1} SHOULD BE THE SAME AS R BECAUSE IF YOU REFLECT TWICE ABOUT THE SAME L, YOU END UP BACK WHERE YOU STARTED



(b) [6 pts] If P is the matrix for orthogonal projection onto L, use the algebraic formula for P to show that $P^2 = P$ (*Hint: Don't forget the assumption that* $w_1^2 + w_2^2 = 1$).

$$P = \frac{\cancel{1}\ 1}{\cancel{w_1^2 + w_2^2}} \begin{pmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{pmatrix} \begin{pmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{pmatrix}$$

$$= \begin{pmatrix} w_1^4 + w_1^2 w_2^2 & w_1^3 w_2 + w_1 w_2^3 \\ w_1^3 w_2 + w_1 w_2^3 & w_1^2 w_2^2 + w_2^4 \end{pmatrix}$$

$$= \begin{pmatrix} w_1^2 (w_1^2 + w_2^2) & w_1 w_2 (w_1^2 + w_2^2) \\ w_1 w_2 (w_1^2 + w_2^2) & w_2^2 (w_1^2 + w_2^2) \end{pmatrix}$$

$$= \begin{pmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{pmatrix} = P \quad \checkmark$$

(c) [4 pts] Using the previous part of the problem, and the formula for R in terms of P, show that $R^2 = I_2$, the 2x2 identity matrix.

$R = 2P - I$

$R^2 = (2P-I)(2P-I)$

$= 4P^2 - 2PI - I2P + I^2$ NOTE $I^2 = I$, $PI = IP = P$

$= 4P^2 - 2P - 2P + I$ PREVIOUS PART OF Q SAYS $P^2 = P$

$= 4P - 2P - 2P + I$

$= I$ ✓

5. True or false (circle your answer; no justification needed).

(a) [2 pts] For two nxn matrices A and B, we have $AB = BA$.

TRUE FALSE

(b) [2 pts] For an invertible nxn matrix A, we have $(A)(A^{-1}) = (A^{-1})(A)$.

TRUE FALSE

(c) [2 pts] For a linear system of n equations with m variables, having nxm coefficient matrix A, to have a unique solution we need $\text{rank}(A) = m$.

TRUE FALSE