1. Consider the matrix

$$A = \begin{pmatrix} 1 & 4 & -2 \\ 1 & 4 & -2 \end{pmatrix} \quad 2 \times 3$$

Recall that A corresponds to a linear transformation T_A.

(a) [2 pts] What are the domain and range of T_A?

range = $\mathrm{im}(T_A)$

domain = $\ker(T_A)$

The domain is $\mathbb{R}^3$

The range is $\mathbb{R}^2$ ✓

$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 1 & 4 & -2 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

image$\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

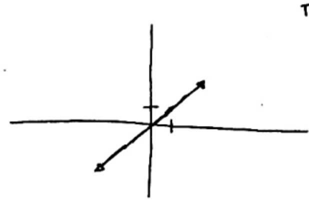
$x_1 = -4x_2 - 2x_3$ span $\left(\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right)$

(b) [2 pts] Describe the image of T_A as a span of vector(s).

$\mathrm{im}(T_A)$ = span (columns)

The image of T_A is span $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ ✓

(c) [4 pts] Describe the image of T_A geometrically. Is it a line? A plane? Draw it.



The image of T_A is the span of the line $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. ✓

2. [6 pts] Is the vector $\vec{b} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$ a linear combination of the vectors $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$? If so, write down the linear combination in the format $\vec{b} = c_1\vec{v} + c_2\vec{w}$. If not, explain why not.

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$c_1\vec{v} + c_2\vec{w} = \vec{b}$

$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \vec{b}$

if $c_2 = -3$, then $c_1 = ?$

$c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + -3 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

$c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -9 \\ -3 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$

if $c_1 = 5$

$\left[\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 1 & 1 & 2 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 3 & -4 \\ 1 & 1 & 2 \\ 0 & 1 & -3 \end{array}\right] \begin{array}{l} -3(\text{III}) \\ -(\text{III}) \\ \\ \end{array}$

$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 1 & 0 & 5 \\ 0 & 1 & -3 \end{array}\right] \begin{array}{l} \\ -\text{I} \\ \\ \end{array}$

$\vec{b} = 5\vec{v} - 3\vec{w}$

$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$

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3. [5 pts] Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \begin{matrix} \\ +\mathrm{I} \\ \\ +3(\mathrm{II}) \\ +\mathrm{II} \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} -2($$

$$\begin{matrix} -2 & 0 & -2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left| \begin{matrix} x_1 = x_3 + x_4 \\ x_2 = -x_4 \end{matrix} \right|$$

Circle all of the following vectors which are members of ker(A).

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{z} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & -3 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 3 & 4 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} \\ +\mathrm{III} \\ \\ +3(\mathrm{II}) \\ \\ \end{matrix}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} \\ \\ -\mathrm{I} \\ -4(\mathrm{II}) \\ \\ \end{matrix}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} -2(\mathrm{II}) \\ \\ +\mathrm{II} \\ \\ \\ \end{matrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left| \begin{matrix} x_1 = x_3 + x_4 \\ x_2 = -x_4 \end{matrix} \right|$$

$$s \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

4. Suppose you know that $A^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & 2 \end{pmatrix}$.

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(a) [6 pts] Find $(AB)^{-1}$.

$(AB)^{-1} = B^{-1}A^{-1}$

$$\begin{pmatrix} -1 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -2+1 & -1+1 \\ 6+2 & 3+2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 8 & 5 \end{pmatrix}$$

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(b) [6 pts] Find B.

$(B^{-1})^{-1} = B$

$$\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \longrightarrow \frac{1}{-2-3}\begin{bmatrix} 2 & -1 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{\cdot}{5} \end{bmatrix}$$

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(c) [2 pts] What was the rank of A? (this should require no computations)

The rank of A is 2. If it is invertible, then the rank ~~of~~ a 2x2 matrix is 2

5. (a) [2 pts] Write down the 2x2 matrix for rotation by an angle θ.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(b) [2 pts] Use the determinant to show that this matrix is invertible.

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The determinant is $ad - bc$

$$\cos^2\theta + \sin^2\theta$$
$$= 1$$

Since the determinant is not zero, the matrix is invertible.

(c) [3 pts] Explain geometrically what the inverse matrix should do, and write the inverse matrix down.

The inverse matrix should rotate, so that the original orientation is returned to. It is a clockwise rotation by angle θ.

$$\frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$ $\cos^2\theta + \sin^2\theta = 1$, so

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$ is the inverse matrix

6. [4 pts] Suppose you know that $\vec{w}$ is in $\ker(B)$, and you also know that $B\vec{v} = \begin{pmatrix}1\\2\\3\end{pmatrix}$. Use this information to find $B(3\vec{w} - 2\vec{v})$.

$B(3\vec{w}-2\vec{v}) = 3B(\vec{w}) - 2B(\vec{v})$

$B(\vec{w}) = 0$ because it is in the kernel of B.

$0 - 2\begin{pmatrix}1\\2\\3\end{pmatrix}$

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$B(3\vec{w}-2\vec{v}) = \begin{pmatrix}-2\\-4\\-6\end{pmatrix}$

7. True or false (circle your answer; no justification needed). In all of the problems below, A is an nxn *square* matrix.

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(a) [2 pts] If A is the coefficient matrix for some linear system, and $\text{rank}(A) = n$, then the system has a unique solution.

TRUE FALSE

(b) [2 pts] If A is the coefficient matrix for some linear system, and $\text{rank}(A) < n$, then the system must have infinitely many solutions.

TRUE FALSE

(c) [2 pts] If A is the coefficient matrix for some linear system, and the system had a unique solution, then the RREF of A must be precisely the identity matrix $I_n = \begin{pmatrix}1 & 0 & 0 & \cdots & 0\\0 & 1 & 0 & & 0\\0 & 0 & 1 & & 0\\\vdots & & & \ddots & \\0 & 0 & 0 & \cdots & 1\end{pmatrix}$.

TRUE FALSE