MATH 33A Midterm 1 Oct. 23, 2017

STUDENT NAME: SOLUTIONS

STUDENT ID NUMBER: __________

DISCUSSION SECTION NUMBER: __________

Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

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1. Consider the matrix

$$A = \begin{pmatrix} 1 & 4 & -2 \\ 1 & 4 & -2 \end{pmatrix}$$

Recall that A corresponds to a linear transformation T_A.

(a) [2 pts] What are the domain and range of T_A?

DOMAIN $= \mathbb{R}^3$

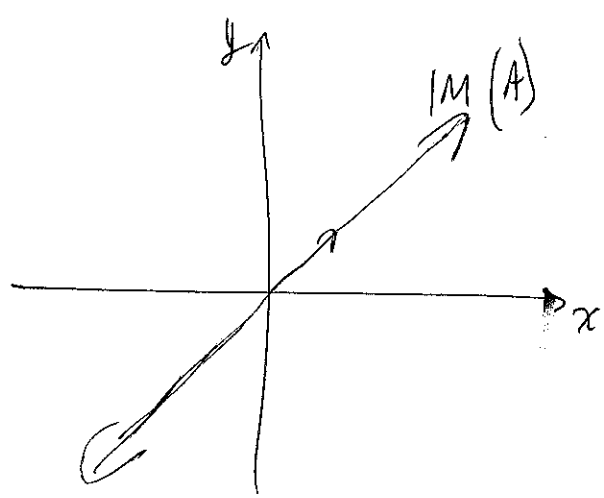
RANGE $= \mathbb{R}^2$

(b) [2 pts] Describe the image of T_A as a span of vector(s).

$\text{IM}(A) = \text{SPAN}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix} \right)$ $\left(\text{ALSO} = \text{SPAN}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right)$

(c) [4 pts] Describe the image of T_A geometrically. Is it a line? A plane? Draw it.

LINE IN $\mathbb{R}^2$ DETERMINED BY $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$



2. [6 pts] Is the vector $\vec{b} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$ a linear combination of the vectors $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$? If so, write down the linear combination in the format $\vec{b} = c_1\vec{v} + c_2\vec{w}$. If not, explain why not.

$$\left(\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 1 & 1 & 2 \end{array}\right) \begin{matrix} \\ \\ -R1 \end{matrix} \longrightarrow \left(\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{array}\right) \begin{matrix} -3R2 \\ \\ +2R2 \end{matrix} \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right)$$

$c_1 = 5$

$c_2 = -3$

So $\begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

3. [5 pts] Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

Circle all of the following vectors which are members of ker(A).

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{z} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$A\vec{v} = \begin{pmatrix} 1+2-1+1 \neq 0 \\ \vdots \\ \end{pmatrix} \neq \vec{0}$$

$$A\vec{w} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \checkmark$$

$$A\vec{x} = \begin{pmatrix} 2+0-2+0 \\ -2+0+2+0 \\ 0 \\ 6+0-6+0 \\ 2+0-2+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \checkmark$$

$$A\vec{y} = \begin{pmatrix} 1+0-2+1 \\ -1+0+2+1 \neq 0 \\ \\ \end{pmatrix} \neq \vec{0}$$

$$A\vec{z} = \begin{pmatrix} 1-2+0+1 \\ -1+0+0+1 \\ 0 \\ 3-4+0+1 \\ 1-1+0+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \checkmark$$

4. Suppose you know that $A^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & 2 \end{pmatrix}$.

(a) [6 pts] Find $(AB)^{-1}$.

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 8 & 5 \end{pmatrix}$$

(b) [6 pts] Find B.

$$B = (B^{-1})^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & 2 \end{pmatrix}^{-1} = \frac{1}{-2-3}\begin{pmatrix} 2 & -1 \\ -3 & -1 \end{pmatrix}$$

$$= \frac{-1}{5}\begin{pmatrix} 2 & -1 \\ -3 & -1 \end{pmatrix}$$

(c) [2 pts] What was the rank of A? (this should require no computations)

RANK$(A) = 2$ (A WAS INVERTIBLE)

5. (a) [2 pts] Write down the 2x2 matrix for rotation by an angle θ.

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(b) [2 pts] Use the determinant to show that this matrix is invertible.

$$\text{Det } A = \cos^2\theta + \sin^2\theta = 1 \neq 0$$

✓

(c) [3 pts] Explain geometrically what the inverse matrix should do, and write the inverse matrix down.

SHOULD ROTATE BY $-\theta$, SO

$$A^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

6. [4 pts] Suppose you know that $\vec{w}$ is in $\ker(B)$, and you also know that $B\vec{v} = \begin{pmatrix}1\\2\\3\end{pmatrix}$. Use this information to find $B(3\vec{w} - 2\vec{v})$.

$$B(3\vec{w} - 2\vec{v}) = 3B\vec{w} - 2B\vec{v} = 3\vec{0} - 2\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$= \begin{pmatrix}-2\\-4\\-6\end{pmatrix}$$

7. True or false (circle your answer; no justification needed). In all of the problems below, A is an *nxn square* matrix.

(a) [2 pts] If A is the coefficient matrix for some linear system, and $\text{rank}(A) = n$, then the system has a unique solution.

TRUE FALSE

(b) [2 pts] If A is the coefficient matrix for some linear system, and $\text{rank}(A) < n$, then the system must have infinitely many solutions.

TRUE FALSE

(c) [2 pts] If A is the coefficient matrix for some linear system, and the system had a unique solution, then the RREF of A must be precisely the identity matrix $I_n = \begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & & 0\\ 0 & 0 & 1 & & 0\\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}$.

TRUE FALSE