1. (10 pts, 2 pts each) Circle **TRUE** or **FALSE** for each statement:

(a) If A is a matrix for which A^2 is defined, then A must be a square matrix.

TRUE **FALSE**

$(m \times n) \cdot (m \times n)$ only works if $m = n$.

(b) If A is a 3×5 matrix, then the image of A is a subspace of $\mathbb{R}^5$.

TRUE **FALSE**

$T(\vec{x}) = A\vec{x}$ is a linear function from $\mathbb{R}^5 \longrightarrow \mathbb{R}^3$, so $\operatorname{im}(A)$ is a subspace of $\mathbb{R}^3$.

(c) If A is a 4×3 matrix and $\operatorname{rank}(A) = 2$, then the system of equations $A\vec{x} = \vec{b}$ must have an infinite set of solutions.

TRUE **FALSE**

It could have no solutions: $\longrightarrow$ $\left[\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & ? \\ 0 & 0 & 0 & 0 \end{array}\right]$

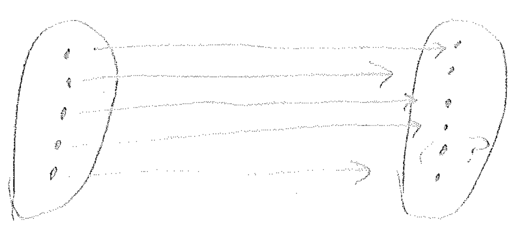
(d) If A is a 5×6 matrix and $\operatorname{rank}(A) = 5$, then the system of equations $A\vec{x} = \vec{b}$ must have an infinite set of solutions.

TRUE **FALSE**

The system must be consistent, and it must have one free variable. $\longrightarrow$ $\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & * & * \\ 0 & 1 & 0 & 0 & 0 & * & * \\ 0 & 0 & 1 & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \end{array}\right]$

(e) If $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible, then the image of T is $\mathbb{R}^n$.

TRUE **FALSE**



A function must "hit" every element of the codomain in order to be invertible. So its image must be the whole codomain.

2. (10 pts) Compute the inverse of the matrix

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 3 & -1 & -1 \\ -2 & 3 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 1 & 0 & 0 \\ 3 & -1 & -1 & 0 & 1 & 0 \\ -2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 3 & -1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} +R3 \\ \cdot(-1) \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ +2\cdot R1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 7 & 1 & 0 & 2 & 3 \end{array}\right] \begin{array}{l} -2\cdot R2 \\ \\ -7\cdot R2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 7 & 2 & 3 \end{array}\right]$$

This is A^{-1}.

$$\boxed{A^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 0 \\ 7 & 2 & 3 \end{bmatrix}}$$

3. (10 pts) Let A be the matrix

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Already in rref! Leading 1's are circled.

(a) (2 pts) What is the rank of A?

$\boxed{3}$ (# of leading 1's)

(b) (4 pts) Completely solve the system of equations $A\vec{x} = \vec{b}$ where

$$\vec{b} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 0 \end{bmatrix}$$

$x_1 - 3x_2 + x_5 = 1$

$x_3 = -2$

$x_4 - x_5 = 4$

Free variables: x_2 and x_5. So let $x_2 = s$, $x_5 = t$.

$$\boxed{\begin{cases} x_1 = 1 + 3s - t \\ x_2 = s \\ x_3 = -2 \\ x_4 = 4 + t \\ x_5 = t \end{cases}}$$

or

$$\boxed{\vec{x} = \begin{bmatrix} 1+3s-t \\ s \\ -2 \\ 4+t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 4 \\ 0 \end{bmatrix} + s\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.}$$

(c) (2 pts) Is there a vector $\vec{b}$ for which the system $A\vec{x} = \vec{b}$ has no solution? If so, give an example. If not, explain why.

Yes. If $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ then there is no solution.

(d) (2 pts) Is there a vector $\vec{b}$ for which the system $A\vec{x} = \vec{b}$ has one unique solution? If so, give an example. If not, explain why.

No. The system has 2 free variables, so it either has no solution or infinitely many.
Or in other words, since rank$(A) <$ # of columns, it may have ∞'ly many solutions. Since rank$(A) <$ # of rows, it may have none.

4. (10 pts) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation for which

$$T(\vec{e}_1) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \quad \text{and} \quad T(\vec{e}_2) = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}.$$

Write down an expression for the matrix of the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ described as follows: first rotate all vectors in the plane by 60° counterclockwise, then project onto the line spanned by $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$, then apply T, and finally make the resulting vector 10 times longer. (You do not need to actually multiply the matrices.)

Rotation by $60°$ CCW: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

Projection along $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$: Use $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$ in $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix} = \begin{bmatrix} 9/25 & -12/25 \\ -12/25 & 16/25 \end{bmatrix}$

Matrix of T: $\begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}$

Scale by factor of 10: $10 I_3 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

$$\boxed{\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 9/25 & -12/25 \\ -12/25 & 16/25 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}}$$

5. (10 pts) Suppose $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+y-z \\ 7x+3y-5z \end{bmatrix}$. What is ker$(T)$?

$$\ker(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \vec{0} \right\}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \vec{0} \iff \begin{cases} 2x+y-z=0 \\ 7x+3y-5z=0 \end{cases}$$ So solve this system.

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 7 & 3 & -5 & 0 \end{array}\right]$$

Note: We can do this without the last column, because it will always be zeros after any row operations.

$$\begin{bmatrix} 7 & 3 & -5 \\ 2 & 1 & -1 \end{bmatrix} -3R_2$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & -1 \end{bmatrix} -2R_1$$

$$\begin{array}{ccc} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix} \\ \uparrow\;\;\uparrow\;\;\uparrow \\ x\;\;\; y\;\;\; z \end{array}$$

z is a free variable, so let $z=t$.

$$\begin{cases} x \quad -2z=0 \\ \quad y+3z=0 \end{cases} \implies \begin{cases} x=2t \\ y=-3t \\ z=t \end{cases}$$

$$\boxed{\text{So } \ker(T) = \left\{ \begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \left\{ t\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\right)}$$