Mathematics 33A - Midterm Examination

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Signature:

- **There are FIVE questions on this examination.**
- **Calculators, notes and books may not be used in this examination.**
- **You may not receive full credit for a correct answer if insufficient work is shown.**

QUESTION	VALUE	SCORE
1	10	10
2	10	10
3	10	10
4	10	10
5	10	10
TOTAL	50	50

1. (10 points) Solve the system of equations given by

$$\begin{cases} x+2y+z=8 \\ x+3y+2z=2 \\ 2x+3y+2z=1. \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 1 & 3 & 2 & 2 \\ 2 & 3 & 2 & 1 \end{array}\right) \xRightarrow[\text{III}-2\text{I}]{\text{II}-\text{I}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 1 & -6 \\ 0 & -1 & 0 & -15 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 15 \\ 0 & 1 & 1 & -6 \end{array}\right) \xRightarrow[\text{III}-\text{II}]{\text{I}-2\text{II}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -22 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 1 & -21 \end{array}\right)$$

$$\xRightarrow{\text{I}-\text{III}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 1 & -21 \end{array}\right)$$

$-22-(-21)$

$-22+21$

$$\boxed{\begin{array}{l} x=-1 \\ y=15 \\ z=-21 \end{array}}$$

$-1+2(15)+(-21)=8$

$-1+3(15)+2(-21)=2$

$2(-1)+3(15)+2(-21)=1$

$$-1\begin{pmatrix}1\\1\\2\end{pmatrix}+15\begin{pmatrix}2\\3\\3\end{pmatrix}-21\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}-1+30-21\\-1+45-42\\-2+45-42\end{pmatrix}=\begin{pmatrix}8\\2\\1\end{pmatrix}\checkmark$$

2. (10 points) Use the standard basis for both $\mathbb{R}^4$ and $\mathbb{R}^3$. Let A equal $\begin{pmatrix} 2 & 4 & 6 & 8 \\ 2 & 2 & 1 & 3 \\ 6 & 8 & 8 & 14 \end{pmatrix}$

and define the linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ by $T(\vec{X}) = A\vec{X}$.

a. Calculate the dimension of the image of T.

$$\begin{pmatrix} 2 & 4 & 6 & 8 \\ 2 & 2 & 1 & 3 \\ 6 & 8 & 8 & 14 \end{pmatrix} \xRightarrow{\frac{1}{2}I} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 3 \\ 6 & 8 & 8 & 14 \end{pmatrix} \xRightarrow[III-6I]{II-2I} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -5 & -5 \\ 0 & -4 & -10 & -10 \end{pmatrix} \xRightarrow{-\frac{1}{2}II} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xRightarrow{I-2II} \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-2\begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix} + \frac{5}{2}\begin{pmatrix} 4 \\ 2 \\ 8 \end{pmatrix} = \begin{pmatrix} -4+10 \\ -4+5 \\ -12+20 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 8 \end{pmatrix} \checkmark$$

$$-1\begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix} + \frac{5}{2}\begin{pmatrix} 4 \\ 2 \\ 8 \end{pmatrix} = \begin{pmatrix} -2+10 \\ -2+5 \\ -6+20 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \\ 14 \end{pmatrix} \checkmark$$

$$\boxed{\mathrm{Dim}(\mathrm{Im}(T)) = 2}$$

✓

b. Calculate the dimension of the kernel T.

$$\begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$x - 2z - w = 0$

$y + \frac{5}{2}z + \frac{5}{2}w = 0$

z

w

$$\boxed{\mathrm{Dim}(\mathrm{Ker}(T)) = 2}$$

✓

$\mathrm{Dim} = \mathrm{Dim}(\mathrm{Im}(T)) + \mathrm{Dim}(\mathrm{Ker}(T))$

$4 = 2 + x$ ✓

$x = 2.$

3. (10 points) Let A equal $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Calculate A^{-1}. You should check your answer to verify that it is correct.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \overset{\mathrm{II}-\mathrm{I}}{\Longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\overset{\mathrm{I}-\mathrm{II}}{\Longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) \overset{\substack{\mathrm{I}-\mathrm{III} \\ \mathrm{II}-\mathrm{III}}}{\Longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\boxed{A^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}}$$

/10

$AA^{-1} = I$

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

4. (10 points) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation whose matrix representation in the standard basis is

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 3 & 2 \\ 2 & 9 & 4 \end{pmatrix}.$$

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a. Find a basis for the kernel of T.

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 3 & 2 \\ 2 & 9 & 4 \end{pmatrix} \xRightarrow{\text{III}-2\text{I}} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 3 & 2 \\ 0 & 3 & 2 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 3 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xRightarrow{\frac{1}{3}\text{II}} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix} \xRightarrow{\text{I}-3\text{II}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{matrix} x - z = 0 \\ y + \frac{2}{3}z = 0 \\ z \end{matrix} \quad \begin{matrix} x = z \\ y = -\frac{2}{3}z \\ z = z \end{matrix} \Rightarrow z\begin{pmatrix} 1 \\ -\frac{2}{3} \\ 1 \end{pmatrix}$$

$$1\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 3 \\ 3 \\ 9 \end{pmatrix} = \begin{pmatrix} -1+2 \\ 0+2 \\ -2+6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \checkmark$$

$$\text{Basis} = \left\{ \begin{pmatrix} 1 \\ -\frac{2}{3} \\ 1 \end{pmatrix} \right\}$$

✓

b. Find a basis for the image of T.

✓

$$\text{Basis} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 9 \end{pmatrix} \right\}$$

5. (10 points) Suppose that β and γ are two bases for $\mathbb{R}^2$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation and

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}.$$

With respect to basis β, the first and second basis vectors of γ are respectively written,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \qquad [I]_\gamma^\beta = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

a. Calculate the change of basis matrix $[I]_\gamma^\beta$.

$$[I]_\gamma^\beta = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_\gamma \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}_\beta \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}_\gamma \mapsto \begin{pmatrix} 2 \\ 3 \end{pmatrix}_\beta$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_\gamma \mapsto \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}_\beta$$

b. Determine $[T]_\gamma^\gamma$.

$$[T]_\gamma^\gamma = [I]_\beta^\gamma [T]_\beta^\beta [I]_\gamma^\beta$$

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$[I]_\beta^\gamma = \left([I]_\gamma^\beta\right)^{-1} = \frac{1}{3-2}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$[T]_\gamma^\gamma = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 11 \\ 7 & 19 \end{pmatrix}$$

$$\begin{array}{r} 33 \\ -38 \end{array}$$

$$\boxed{[T]_\gamma^\gamma = \begin{pmatrix} -2 & -5 \\ 3 & 8 \end{pmatrix}}$$

$$\begin{tikzcd}
\vec{x}_\beta \arrow[r, "A"] \arrow[d, shift left, "S^{-1}"] & T(\vec{x})_\beta \arrow[d, shift left, "S^{-1}"] \\
{[\vec{x}]_\gamma} \arrow[r, "B"] \arrow[u, shift left, "S"] & {[T(\vec{x})]_\gamma} \arrow[u, shift left, "S"]
\end{tikzcd}$$

$A = [T]_\beta^\beta \qquad B = ?$

$B = S^{-1}AS \qquad S = [I]_\gamma^\beta$