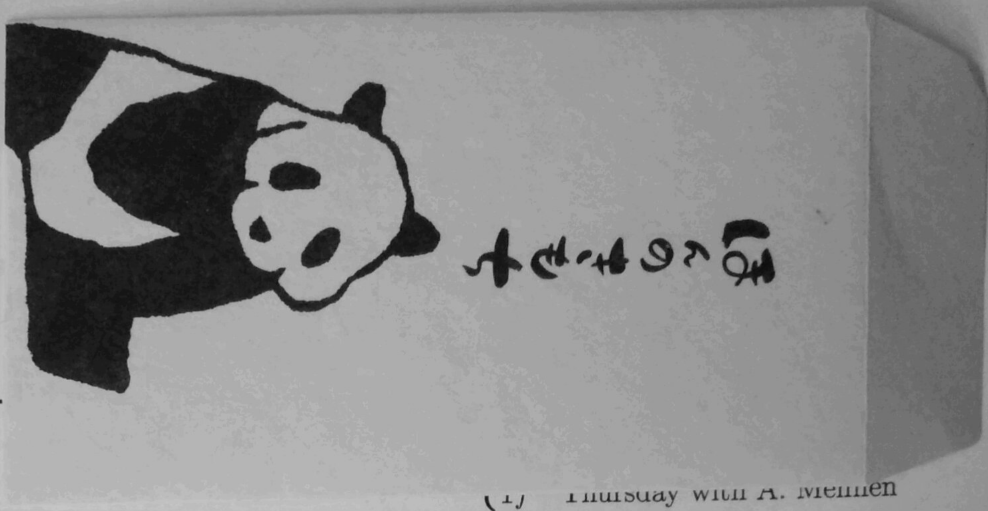
First Name:

Last Name:

Section: 1



(1) Thursday with A. Mennen

Rules.

- There are **FOUR** problems, **each worth 10 points**.
- No calculators, computers, notes, books, e.t.c....
- Use the backs of the pages. There is one spare page at the back.
- Out of consideration for your class-mates, no chewing, humming, pen-twirling, or snoring. Try to sit still.
- Turn off your cell-phone.

1	2	3	4	Σ
6	8	7	10	31

(1) (a) Find the QR factorization of the matrix

6

$$A = \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \shortparallel & \shortparallel & \shortparallel \end{array}$$
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

0 (b) Find the point in im(A) that is closest to the point $P = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

A) $Q = |\vec{u}_1\ \vec{u}_2\ \vec{u}_3|$

$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{2}\begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix}$ $\quad \vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_1)\vec{u}_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix} - (\frac{1}{2} \cdot \cancel{2})\frac{1}{2}\begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 \\ -1 \\ -1 \\ 1 \end{vmatrix}$ $\quad \vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp} = \frac{1}{1}\vec{v}_2^{\perp}$

$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 = \begin{vmatrix} 0 \\ 2 \\ 1 \\ -1 \end{vmatrix} + (\frac{1}{2}(+4))\frac{1}{2}\begin{vmatrix} 1 \\ -1 \\ -1 \\ 1 \end{vmatrix} - (\frac{1}{2}(2))\frac{1}{2}\begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ 2 \\ 1 \\ -1 \end{vmatrix} + \begin{vmatrix} 1 \\ -1 \\ -1 \\ 1 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 \\ 1 \\ -1 \\ -1 \end{vmatrix}$ $\quad \vec{u}_3 = \frac{1}{\|\vec{v}_3^{\perp}\|}\vec{v}_3^{\perp} = \frac{1}{1}\vec{v}_3^{\perp}$

$Q = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix}$ $\quad R = \begin{vmatrix} \|\vec{v}_1\| & \vec{u}_1 \cdot \vec{v}_2 & \vec{u}_1 \cdot \vec{v}_3 \\ 0 & \|\vec{v}_2^{\perp}\| & \vec{u}_2 \cdot \vec{v}_3 \\ 0 & 0 & \|\vec{v}_3^{\perp}\| \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$

$$\boxed{A = QR = \underbrace{\begin{vmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \end{vmatrix}}_{Q} \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix} \Leftarrow R}$$

B) Find $\text{proj}_A\vec{P} \rightsquigarrow QP$ where $Q = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{vmatrix}$

going $\mathbb{R}^4 \to \mathbb{R}^3$ collapses a dimension (rows?)

$QP = \frac{1}{2}\begin{vmatrix} 6 \\ 2 \\ -4 \\ 0 \end{vmatrix} = \boxed{\begin{vmatrix} 3 \\ 1 \\ -2 \\ 0 \end{vmatrix}}$ ×

8

(2) Find a line $y = cx + d$ best fitting the data points $(0, 5)$, $(1, 0)$, and $(2, 3)$.

$$\underset{A}{\begin{vmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{vmatrix}} \underset{\vec{v}}{\begin{vmatrix} c \\ d \end{vmatrix}} = \underset{\vec{b}}{\begin{vmatrix} 5 \\ 0 \\ 3 \end{vmatrix}}$$

$$A\vec{v} = \vec{b}$$
$$A\vec{v} - \vec{b} = 0$$
$$A^T(A\vec{v} - \vec{b}) = 0$$
$$A^TA\vec{v} = A^Tb$$

$$A^T = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} \quad A^TA = \begin{vmatrix} 3 & 3 \\ 3 & 5 \end{vmatrix} \quad A^T\vec{b} = \begin{vmatrix} 8 \\ 6 \end{vmatrix}$$

$$\text{rref}(A^TA \,\vdots\, A^Tb) = \text{rref}\left|\begin{array}{cc:c} 3 & 3 & 8 \\ 3 & 5 & 6 \end{array}\right|_{-R_1} = \text{rref}\left|\begin{array}{cc:c} 3 & 3 & 8 \\ 0 & 2 & -2 \end{array}\right|^{-\frac{3}{2}R_2}_{\times(\frac{1}{2})}$$

$$= \text{rref}\left|\begin{array}{cc:c} 3 & 0 & 11 \\ 0 & 1 & -1 \end{array}\right|^{\times(\frac{1}{3})}$$

$$= \text{rref}\left|\begin{array}{cc:c} 1 & 0 & 11/3 \\ 0 & 1 & -1 \end{array}\right|$$

Closest line is $y = \frac{11}{3}c - 1$ $\quad \left.\begin{array}{l} c = 11/3 \\ d = -1 \end{array}\right\}$

(3) (a) Find all possible solutions to

$$\begin{bmatrix}1 & 1 & 2\\ 1 & 2 & 3\end{bmatrix}\vec{x}=\begin{bmatrix}1\\ 0\end{bmatrix} \qquad \vec{x}=\begin{vmatrix}x_1\\ x_2\\ x_3\end{vmatrix}$$

(b) Find the solution $\vec{x}$ to the linear system in part (a) that minimizes $\|\vec{x}\|$.

A) $\text{rref}\begin{vmatrix}1 & 1 & 2 & \vdots & 1\\ 1 & 2 & 3 & \vdots & 0\end{vmatrix}_{-R_1} = \text{rref}\begin{vmatrix}1 & 1 & 2 & \vdots & 1\\ 0 & 1 & 1 & \vdots & -1\end{vmatrix}^{-R_2} = \text{rref}\begin{vmatrix}1 & 0 & 1 & \vdots & 2\\ 0 & 1 & 1 & \vdots & -1\end{vmatrix} \Rightarrow \begin{cases}x_1+x_3=2\\ x_2+x_3=-1\end{cases}$

$x_1=2-x_3$
$x_2=-1-x_3$

let $x_3=r$ $\quad \vec{x}=\begin{vmatrix}2-r\\ -1-r\\ r\end{vmatrix}$? $\text{span}\left(\begin{vmatrix}-1\\ 2\\ 0\end{vmatrix}, \begin{vmatrix}-1\\ -1\\ 1\end{vmatrix}\right)$ No!

B) Want $A\vec{x}-\vec{b}=0$

$A^T(A\vec{x}-\vec{b})=0$

$A^TA\vec{x}=A^T\vec{b}$

$A^T=\begin{vmatrix}1 & 1\\ 1 & 2\\ 2 & 3\end{vmatrix}$ $\quad A^TA=\begin{vmatrix}2 & 3 & 5\\ 3 & 5 & 8\\ 5 & 8 & 13\end{vmatrix}$ $\quad A^T\vec{b}=\begin{vmatrix}1\\ 1\\ 2\end{vmatrix}$

$A^TA\vec{x}=A^T\vec{b}$ $\quad x_1=\dfrac{\det((A^TA)_b)}{\det(A^TA)}$

$\vec{x}=\begin{vmatrix}x_1\\ x_2\\ x_3\end{vmatrix}$

span of $\vec{x}$

$A=\begin{vmatrix}-1 & -1\\ 2 & -1\\ 0 & 1\end{vmatrix}$ $\quad b=\begin{vmatrix}1\\ 0\end{vmatrix}$ $\quad A^T=\begin{vmatrix}-1 & 2 & 0\\ -1 & -1 & 0\end{vmatrix}$

$A\vec{x}=\vec{b}$ $\quad A^TA=\begin{vmatrix}5 & -1\\ -1 & 3\end{vmatrix}$ $\quad A^T\vec{b}=\begin{vmatrix}-1\\ -1\end{vmatrix}$

$\begin{vmatrix}5 & -1\\ -1 & 3\end{vmatrix}\vec{x}=\begin{vmatrix}-1\\ -1\end{vmatrix}$

$\text{rref}(A^TA \vdots A^T\vec{b}) = \text{rref}\begin{vmatrix}5 & -1 & \vdots & -1\\ -1 & 3 & \vdots & -1\end{vmatrix} = \text{rref}\begin{vmatrix}1 & -3 & \vdots & 1\\ 5 & -1 & \vdots & -1\end{vmatrix}_{-5R_1}$

$= \text{rref}\begin{vmatrix}1 & -3 & \vdots & 1\\ 0 & 14 & \vdots & -6\end{vmatrix}^{+3R_2'}_{\times(\frac{1}{14})}$

$= \text{rref}\begin{vmatrix}1 & 0 & \vdots & -2/7\\ 0 & 1 & \vdots & -3/7\end{vmatrix}$

$\left.\begin{matrix}x_1=-2/7\\ x_2=-3/7\end{matrix}\right\}$

Solution to minimize $\|\vec{x}\|$ is $\vec{x}=\begin{vmatrix}-2/7\\ -3/7\end{vmatrix}$

(4) (a) Compute the determinant of

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 5 & 0 & 2 & 3 & 4 \\ 5 & 0 & 0 & 3 & 4 \\ 6 & 1 & 2 & 5 & 7 \\ 6 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Is the matrix A invertible? YES / NO

A $\det \begin{vmatrix} 1 & 1 & 2 & 3 & 4 \\ 5 & 0 & 2 & 3 & 4 \\ 5 & 0 & 0 & 3 & 4 \\ 6 & 1 & 2 & 5 & 7 \\ 6 & 0 & 0 & 0 & 1 \end{vmatrix} = (-1)^6(6)\det \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 1 & 2 & 5 & 7 \end{vmatrix}_{-R_1-\frac{2}{3}R_3} + (-1)^{10}(1)\det \begin{vmatrix} 1 & 1 & 2 & 3 \\ 5 & 0 & 2 & 3 \\ 5 & 0 & 0 & 3 \\ 6 & 2 & 2 & 5 \end{vmatrix}$ $\leftrightarrow R_3$ $\leftrightarrow R_4$

columns not linearly independent so the determinant is 0!

$= 6\det \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 2/3 \end{vmatrix} + (-1)^{3^{\vee}}\det \begin{vmatrix} 5 & 2 & 3 \\ 5 & 0 & 3 \\ 6 & 2 & 5 \end{vmatrix} + (-1)^6(1)\det \begin{vmatrix} 1 & 2 & 3 \\ 5 & 2 & 3 \\ 5 & 0 & 3 \end{vmatrix}$

$= 6(1 \cdot 2 \cdot 3 \cdot \frac{1}{3}) - \left((-1)^{3^{\vee}}\det \begin{vmatrix} 5 & 3 \\ 6 & 5 \end{vmatrix} + (-1)^5(2)\det \begin{vmatrix} 5 & 3 \\ 5 & 3 \end{vmatrix}\right) + (-1)^3(2)\det \begin{vmatrix} 5 & 3 \\ 5 & 3 \end{vmatrix} + (-1)^4(2)\det \begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix}$

$= 12 - (-2(25-18)) + (2(3-15)) = 12 + 14 - 24 = \boxed{2}$ ✓

B) Yes, $\det|A| \neq 0$ ✓