MATH 33A: MidTerm II

Name: Kai

Signature:

SID number:

Question #0: What is the most interesting thing you have learned in this class since the last exam?

Instructions:
Show all work to receive full credit. Feel free to use the back of each paper *but please indicate that you have done so.* No calculators or formula sheets allowed and please box your answers. **You will receive 1 extra credit point for filling out all the information above.**

Problem	Possible	Score
1	20	
2	22	
3	16	
4	20	
5	22	
EC	1	
Total	100	

Problem 1: Determine wisely (20 pts)

Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 2 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 5 \\ 3 & 4 & 5 & 8 & 5 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 9 \\ 3 \\ 6 \\ 0 \\ 3 \end{bmatrix}$$

- Find $\det(A)$.
- Let $\vec{x}$ be the solution to $A\vec{x} = \vec{b}$. Find x_5 (the fifth entry of $\vec{x}$).

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 2 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 5 \\ 3 & 4 & 5 & 8 & 5 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 2 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 5 \\ 3 & 4 & 5 & 8 & 5 \end{bmatrix}$$

$$P_1 \qquad\qquad\qquad P_2$$

so $\det(A) = \operatorname{sign}(P_1)\operatorname{Prod}(P_1) + \operatorname{sgn}(P_2)\operatorname{Prod}(P_2)$

$= (-1)^5 \cdot 3 \cdot 1 \cdot 1 \cdot 2 \cdot 5 + (-1)^6 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 5 = -30 + 120 = 90$

- By Cramers rule

$$x_5 = \frac{\det(A_{b,5})}{\det A} = \frac{\det \begin{bmatrix} 0 & 1 & 2 & 3 & 9 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 2 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 8 & 3 \end{bmatrix}}{90} = 0$$

since $A_{b,5}$ has a row of zeros.

Problem 2: Hermite Polynomials (22 pts)

The Hermite polynomials h_k are defined as:

$$h_k = e^{-x^2} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{x^2}$$

and are solutions to Schrodinger's wave equation. The first three can be computed to be $h_0 = 1,\ h_1 = 2x,\ h_2 = 2 + 4x^2$.

- Let $\mathbb{B} = \{h_0,\ h_1,\ h_2\}$ be a basis for P_2 and $f(x) = ax^2 + bx + c$ be an element in P_2. Find $[f(x)]_{\mathbb{B}}$.
- Find the $\mathbb{B}$-matrix for the linear transformation $T : P_2 \to P_2$ defined as

$$T(f) = 2f' - f$$

- Is T an isomorphism? Explain why or why not.

- $f(x) = ax^2 + bx + c = \frac{a}{4}(4x^2+2) + \frac{b}{2}(2x) + (c - \frac{a}{2})1$

$= \frac{a}{4}h_2 + \frac{b}{2}h_1 + (c - \frac{a}{2})h_0$

so $[f(x)]_\beta = \begin{bmatrix} c - a/2 \\ b/2 \\ a/4 \end{bmatrix}$

- $B = \left[[T(h_0)]_\beta \ [T(h_1)]_\beta \ [T(h_2)]_\beta \right]$

$[T(h_0)]_\beta = [0 - 1]_\beta = [-1]_\beta = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$[T(h_1)]_\beta = [4 - 2x]_\beta = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$

$[T(h_2)]_\beta = [16x - 2 - 4x^2]_\beta = \begin{bmatrix} 0 \\ 8 \\ -1 \end{bmatrix}$

so $B = \begin{bmatrix} -1 & 4 & 0 \\ 0 & -1 & 8 \\ 0 & 0 & -1 \end{bmatrix}$

- T is an isomorphism because B is invertible.

Problem 3: Factor madness (16pts)

Please find the QR factorization of the matrix:

$$M = \begin{bmatrix} 1 & -1 \\ -2 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$$

$M = QR$.

$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}. \quad \vec{u}_1 = \frac{1}{\sqrt{1^2+2^2+1^2}} \vec{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \frac{\sqrt{6}}{6} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{\sqrt{6}}\left(\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right) \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

$= -\frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \|\vec{v}_2^{\perp}\| = \frac{2\sqrt{3}}{3}$

$\vec{u}_2 = -\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = -\frac{\sqrt{3}}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$

$\vec{u}_1 \cdot \vec{v}_2 = \frac{\sqrt{6}}{6} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = -\frac{\sqrt{6}}{3}.$ Thus

$$\underbrace{\begin{bmatrix} 1 & -1 \\ -2 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \\ -\frac{2\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \\ 0 & 0 \\ \frac{\sqrt{6}}{6} & +\frac{\sqrt{3}}{3} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{6} & -\frac{\sqrt{6}}{3} \\ 0 & \frac{2\sqrt{3}}{3} \end{bmatrix}}_{R}$$

Problem 4: How fitting (20 pts)

Consider the three data points $(a_1, b_1) = (-1, 0)$, $(a_2, b_2) = (0, 1)$, $(a_3, b_3) = (1, -2)$.

- Find a least squares fit of this data to a linear function $f(x) = c_0 + c_1 x$.
- Please compute the error associated to your fit.

a) $f(x) = c_0 + c_1 x \Rightarrow \begin{array}{l} c_0 - c_1 = 0 \\ c_0 = 1 \\ c_0 + c_1 = -2 \end{array}$

$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \Rightarrow A \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \vec{b}$

$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}^* = (A^T A)^{-1} A^T \vec{b}$, $A^T A = \left(\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$(A^T A)^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$. thus.

$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}^* = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1 \end{bmatrix}$ ✓

$\Rightarrow \boxed{f(x) = -\frac{1}{3} - x.}$

b) Error is $\|\vec{b} - Ac^*\|$, $\vec{b} - Ac^* = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1/3 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} -2/3 \\ 4/3 \\ -2/3 \end{bmatrix}$

So $\|\vec{b} - Ac^*\| = \left\| \begin{bmatrix} -2/3 \\ 4/3 \\ -2/3 \end{bmatrix} \right\| = \frac{2\sqrt{6}}{3}$

Problem 5: Short answer (22 pts)

Please circle your answer.

- What is the dimension of the space of all lower triangular 3×3 matrices? $\boxed{6}$
- True or False: There exists an isomorphism from P_3 to $\mathbb{C}$.
- True or False: If A is an orthogonal matrix and B is a rotation matrix then BA is a rotational matrix.
- True or False: If A is an orthogonal matrix and B is a rotation matrix then AB is an orthogonal matrix.
- True or False: If T is a linear transformation from C^∞ to C^∞ then the intersection of $im(T)$ and $ker(T)$ is $\{0\}$.
- Let Ω be any subset of $\mathbb{R}^2$ with nonzero area. How much bigger is the area of $T(\Omega)$, as compared to the area of Ω, if $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$? $\boxed{\text{3 times bigger}}$
- True or False: There exists invertible 2×2 matrices A and B such that $\det(A + B) = \det(A) + \det(B)$.
- True or False: If A is invertible then $A\texttt{adj}(A) = \texttt{adj}(A)A$.
- True or False: Let A and B be $n \times n$ matrices. If A is orthogonal and B is non-invertible then $\det(AB) = \det(BA)$.
- Suppose A is an $n \times n$ matrix such that $A = A^{-1}$. Find all the possible values of $\det(A)$. $\boxed{\det(A) = \pm 1}$
- True or False: Every invertible matrix A can be expressed as a product of an orthogonal matrix and an upper triangular matrix.