MATH 33A: MidTerm I

Name: Key

Signature:

SID number:

What is the most interesting thing you have learned in this class so far?

Circle your section:

1A (Justin Shih, T 9am) 1B (Justin Shih, R 9am)
1C (Joshua Baron, T 9am) 1D (Joshua Baron,R 9am)
1E (Judah Jacobson T 9am) 1F (Judah Jacobson R 9am)

Instructions:
Show all work to receive full credit and feel free to use the back of each paper. No calculators, references, or formula sheets allowed and please box your answers. You will receive 1 extra credit point for filling out all the information above.

Problem	Possible	Score
1	15	
2	12	
3	21	
4	22	
5	10	
6	20	
EC	1	
Total	100	

Problem 1 (15 pts)

Let A be an $n \times n$ matrix. List four equivalent statements to:
"The column vectors of A form a basis of $\mathbb{R}^n$."

Any of the precisely worded statements in Summary 3.3.10 on pg. 133.

Let A be an $n \times n$ matrix, B be an $m \times n$ matrix, and C be an $n \times m$ matrix, $\vec{v}$ be an $n \times 1$ vector and $\vec{u}$ be an $1 \times n$ vector, where $n \neq m$. Which of the following expressions are well defined? No work is need here for full credit.

1. C^3 Not well defined
2. $\vec{v} \cdot \vec{u}$ Well defined
3. ABC Not well defined
4. $\vec{u}A$ Well defined
5. $B\vec{v}$ Well defined
6. $CBA\vec{u}$ Not well defined
7. $\vec{u} \cdot \vec{v}$ Well defined

Problem 2 (12 pts)

If possible, find the inverse of

$$A = \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix}$$

what is the rank of A?

$$\left[\begin{array}{cc|cc} -1 & -2 & 1 & 0 \\ -2 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} -3 & 0 & 1 & 1 \\ -2 & 2 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & -1/3 & -1/3 \\ -1 & 1 & 0 & 1/2 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -1/3 & -1/3 \\ 0 & 1 & -1/3 & 1/6 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1/3 & -1/3 \\ -1/3 & 1/6 \end{bmatrix}$$

Since A is invertible, $\text{rank}(A) = 2$.

Problem 3 (21 pts)

Part I: Please construct a 2×2 matrix A which rotates a vector by $90°$ *clockwise.* Next construct a matrix C which projects all vectors in $\mathbb{R}^2$ onto the line $y = 2x$.

Rotation by angle θ is given by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Thus for $\theta = -90°$ we get

$$A = \begin{bmatrix} \cos(-90) & -\sin(-90) \\ \sin(-90) & \cos(-90) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

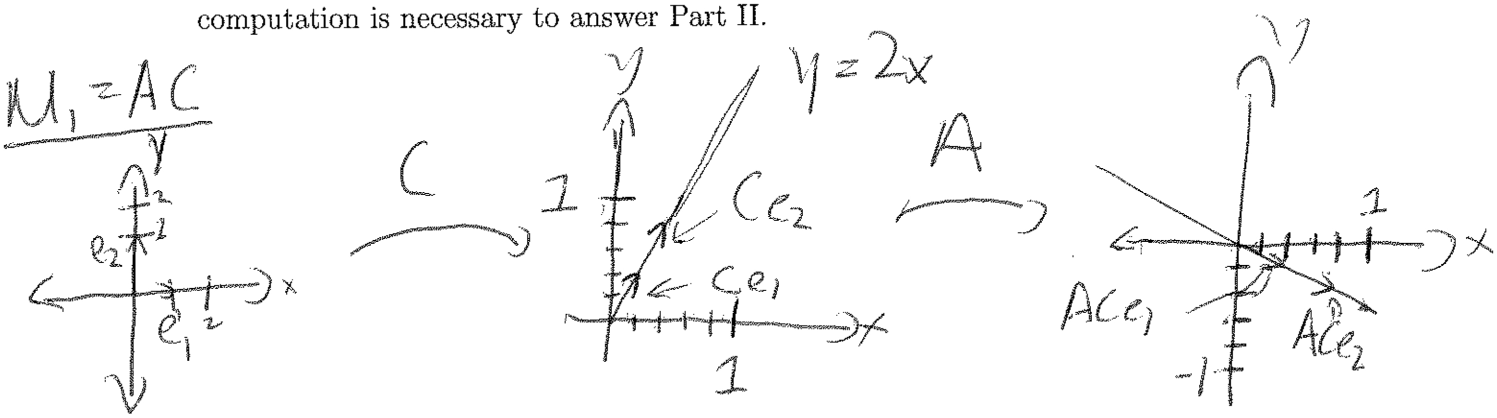
The line $y = 2x$ is spanned by $\vec{W} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

thus

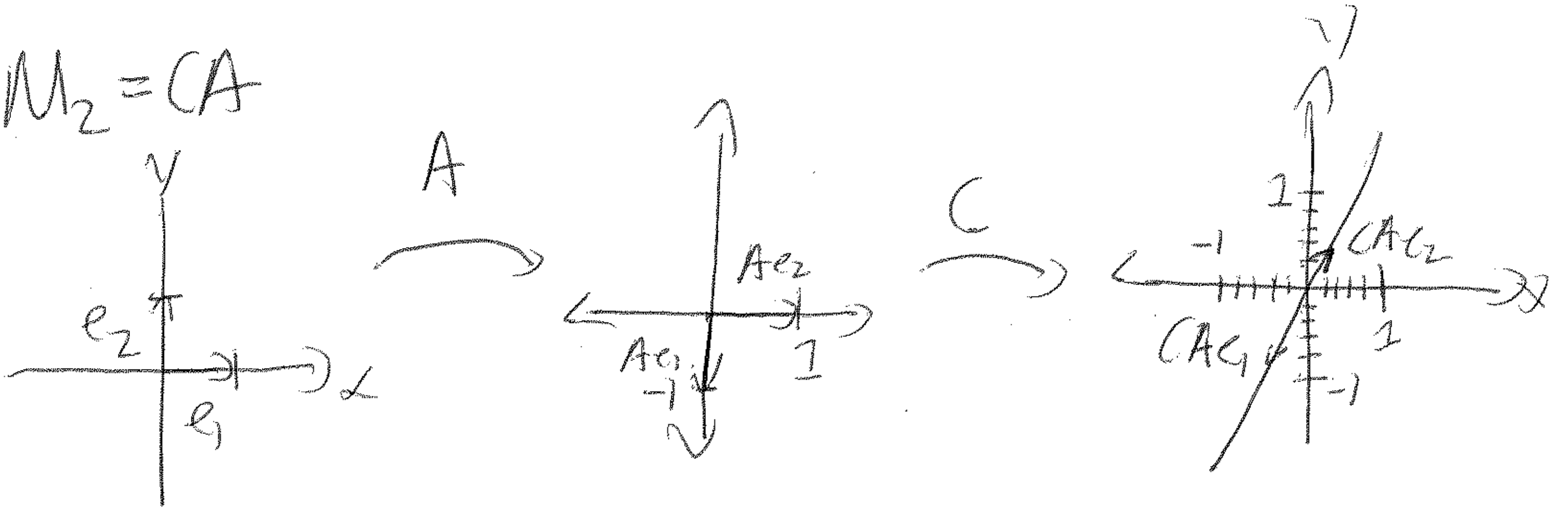
$$C = \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}$$

Part II: Carefully plot and label what happens geometrically to $\vec{e}_1$ and $\vec{e}_2$ under the linear transformations $M_1 = AC$ and $M_2 = CA$ i.e. plot $M_1(\vec{e}_i)$ and $M_2(\vec{e}_i)$ for $i = 1, 2$. What can we conclude about the commutativity of A and C? Note: No computation is necessary to answer Part II.

$\underline{M_1 = AC}$



$M_2 = CA$



$AC \neq CA.$

Problem 4 (22pts)

Let A be

$$A = \begin{bmatrix} 3 & 6 & -3 & 6 & -6 \\ 0 & 0 & 2 & -6 & 4 \end{bmatrix}.$$

1. What is the domain and range of the linear transformation $T(\vec{x}) = A\vec{x}$?
2. Find rref(A).
3. Find a basis for Im(A).
4. Find a basis for Ker(A).

(1) Domain is $\mathbb{R}^5$, Range is $\mathbb{R}^2$.

(2) $\begin{bmatrix} 3 & 6 & -3 & 6 & -6 \\ 0 & 0 & 2 & -6 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 2 & -2 \\ 0 & 0 & 1 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 & 2 \end{bmatrix}$

$= \text{rref}(A)$.

(3) basis for Im(A), Well Im(A) = span $\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right)$

and since $\begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ are L.I. They are a basis.

(4) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underset{v_1}{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} s + \underset{v_2}{\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}} t + \underset{v_3}{\begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}} r$

$\{v_1, v_2, v_3\}$ are a basis for Ker (A).

Problem 5 (10pts)

Consider the plane $2x_1 - 3x_2 + 4x_3 = 0$ with basis $\mathbb{B}$ vectors $\vec{v_1}$ and $\vec{v_2}$ where only v_1 is known:

$$\vec{v_1} = \begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix}.$$

Find the second basis vector v_2 which satisfies

$$\begin{bmatrix} 11 \\ 6 \\ -1 \end{bmatrix}_{\mathbb{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Well $\begin{bmatrix} 11 \\ 6 \\ -1 \end{bmatrix}_{\beta} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow S\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \\ -1 \end{bmatrix}$

when $S = [V_1 \; V_2]$. Thus

$$\begin{bmatrix} 8 & a \\ 4 & b \\ -1 & c \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \\ -1 \end{bmatrix} \Rightarrow \begin{matrix} a = 5 \\ b = 2 \\ c = 1 \end{matrix}$$

so $V_2 = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$

Problem 6 (20pts)

Please state whether the following statements are necessarily true (no work needs to be shown here).

1. A vertical shear linear transformation is always invertible. True
2. If V and W are subspaces of $\mathbb{R}^n$ such that $\dim(V) + \dim(W) = n$, then $V \cup W = \mathbb{R}^n$. False
3. There exists a 2×2 matrix $A \neq 0$ such that $A^2 = 0$, where "0" is the 2×2 matrix of zeros. True
4. Let the line L be any line through the origin. Then you can write any vector $\vec{x} \in \mathbb{R}^2$ as $\vec{x} = \text{proj}_L(\vec{x}) + \text{ref}_L(\vec{x})$. False
5. The domain of the linear transformation $T(x) = Ax$, where A is a 20×10 matrix, is $\mathbb{R}^{20}$. False
6. There is a 3×3 invertible matrix where seven of the nine entries are 1. True
7. If A is an 5×5 matrix and you can find 3 distinct solutions to $Ax = 0$ then you can always find 4. True
8. A 3×2 matrix is never invertible. True
9. If A is a $r \times p$ matrix then $\dim(\text{Im}(A))$ $+\dim(\ker(A))$ = p. True
10. If the set of vectors $\{v_1 \dots v_k\}$ spans $\mathbb{R}^n$, then $k = n$. False