Mathematics 33A Exam #2 Fall Quarter 2016

[redacted]

Instructions:

- You have 50 minutes for this exam.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers. If you use the back of a page please indicate that you have work on the reverse side.
- You may not use books, notes, calculators, mobile phones, or any outside help during this exam. You may not collaborate with other students in any way during this exam.

Problem	Points	Score
1	10	6
2	10	7
3	10	10
4	10	10
Total	40	33

1. True or False questions (10 points total); circle 'T' if the item is true or 'F' if the item is false.

a. (2 points) If the matrix A represents a rotation through $\pi/2$ and the matrix B a rotation through $\pi/4$, then A and B are similar.

$A = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$

$B = \begin{bmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{bmatrix}$

~~T~~ F

b. (2 points) If A and B are similar, and A is invertible, then B must be invertible as well.

$AS = SB$ $\quad AA^{-1} = I_n$ $\quad BB^{-1} = S^{-1}S$

$A = SBS^{-1}$ $\quad SBS(SBS)^{-1} = I_n$ $\quad BB^{-1} = I_n$

$SB\cancel{SS^{-1}}(SB)^{-1} = I_n$

$SBB^{-1}S^{-1} = I_n$

T F

c. (2 points) If A is an invertible matrix, then the equation $(A^{-1})^T = (A^T)^{-1}$ must hold.

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{bmatrix}$

T F

d. (2 points) For all 4×4 matrices A, $\det(4A) = 4\det A$.

$\begin{bmatrix} -c & -d \\ a & b \end{bmatrix} = \frac{1}{\sqrt2}\begin{bmatrix} a+b & - \\ c+d & - \end{bmatrix}$

T F

$d = a - b$ $\quad a = \frac{1}{\sqrt2}+b$ $\quad -c = a+b$

$d = \frac{1}{\sqrt2}a$ $\quad b = \left(1 - \frac{1}{\sqrt2}\right)a$ $\quad c = \left(\frac{1}{\sqrt2} - 2\right)a$

$\frac{1}{\sqrt2} - \left(\frac{1}{\sqrt2} - 2 - \frac{1}{2} + \frac{2}{\sqrt2}\right) =$

e. (2 points) If A is an $n \times n$ matrix such that $A = SBS^{-1}$, then $\det A^5 = \det B^5$.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$

~~T~~ F

$A^5 = (S^{-1}BS^{-1})(S^{-1}BS^{-1})(S^{-1}BS^{-1})(S^{-1}BS^{-1})(S^{-1}BS^{-1})$

$A^5 = S^{-1}B^5S^{1}$

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2. (10 points) Suppose that

$$T(\vec{x}) = A\vec{x} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\vec{x}$$

is a linear transformation expressed as the matrix A with respect to the standard basis $\mathcal{S} = (\vec{e}_1, \vec{e}_2)$. Let $\mathcal{B} = (\vec{v}_1, \vec{v}_2) = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right)$ be another basis for $\mathbb{R}^2$.

a. (7 points) Find the matrix $B = [T]_\mathcal{B}$, which expresses the linear transformation T as a matrix with respect to the basis $\mathcal{B}$.

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$$B = \begin{bmatrix} [T(\cancel{\vec{e_1}})]_{\mathcal{B}} & [T(\cancel{\vec{e_2}})]_{\mathcal{B}} \end{bmatrix} \quad v_1 \quad v_2$$

$$T(\vec{e_1}) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 4 & -2 \end{array}\right] -r_1$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 2 & -2 \end{array}\right] \div 2$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1 \end{array}\right] -2r_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

$$[T(\vec{e_1})]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$T(\vec{e_2}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 4 & 3 \end{array}\right] -r_1$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 2 & 2 \end{array}\right] \div 2$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \end{array}\right] -2r_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right]$$

$$[T(\vec{e_2})]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\boxed{B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}$$

b. (3 points) Find A^5. (Hint: your work in part 'a.' might be useful.)

$$A^2 = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix} = \begin{bmatrix} -14 & 15 \\ -30 & 31 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} -14 & 15 \\ -30 & 31 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -30 & 31 \\ -62 & 63 \end{bmatrix}$$

$$\boxed{A^5 = \begin{bmatrix} -30 & 31 \\ -62 & 63 \end{bmatrix}}$$

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3. (10 points total) Let $A = \begin{array}{c} \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \\ \begin{pmatrix} 3 & 3 & 0 \\ 4 & 4 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{pmatrix} \end{array}$.

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a. (7 points) Find the QR-decomposition of A. Make sure to show your work.

$r_{11} = \|\vec{v}_1\| = 5, \quad \vec{u}_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$

$r_{12} = \vec{u}_1 \cdot \vec{v}_2 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 3 \\ 4 \end{bmatrix} = 5, \quad \vec{v}_2^{\perp} = \begin{bmatrix} 3 \\ 4 \\ 3 \\ 4 \end{bmatrix} - 5 \cdot \frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}, \quad r_{22} = \|\vec{v}_2^{\perp}\| = 5, \quad \vec{u}_2 = \frac{1}{5}\begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$

$r_{13} = \vec{u}_1 \cdot \vec{v}_3 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix} = 0, \quad r_{23} = \vec{u}_2 \cdot \vec{v}_3 = \frac{1}{5}\begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix} = 0$

$\vec{v}_3^{\perp} = \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix} - 0 - 0 = \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix}, \quad r_{33} = \|\vec{v}_3^{\perp}\| = 5, \quad \vec{u}_3 = \frac{1}{5}\begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix}$

$$\boxed{A = \frac{1}{5}\begin{bmatrix} 3 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 5 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}}$$

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b. (3 points) Find the matrix which represents orthogonal projection onto the subspace of $\mathbb{R}^4$ which is defined by the columns of A.

$\cancel{P} = QQ^T$

$P = \frac{1}{25}\begin{bmatrix} 3 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & -4 & 3 \end{bmatrix}$

$= \frac{1}{25}\begin{bmatrix} 9 & 12 & 0 & 0 \\ 12 & 16 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 25 \end{bmatrix}$

$$\boxed{P = \begin{bmatrix} \frac{9}{25} & \frac{12}{25} & 0 & 0 \\ \frac{12}{25} & \frac{16}{25} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

4. (10 points) Find the equation of the line which best fits, in the least squares sense, the following points in the 'x, y'-plane: $(-1,-1),(0,1),(1,4)$.

$y = mx + b$

$-1 = -m + b$

$1 = 0m + b$

$4 = 1m + b$

$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$

$A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ rank $= 2$ $(A^TA)^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$

$= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$

$\vec{x}^* = \begin{bmatrix} \frac{5}{2} \\ \frac{4}{3} \end{bmatrix} = \begin{bmatrix} m \\ b \end{bmatrix}$

$y = mx + b$

$$\boxed{y = \frac{5}{2}x + \frac{4}{3}}$$