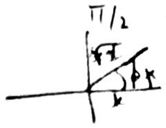
$\pi/2$



1. True or False questions (10 points total); circle 'T' if the item is true or 'F' if the item is false.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

a. (2 points) If the matrix A represents a rotation through $\pi/2$ and the matrix B a rotation through $\pi/4$, then A and B are similar.

$$\begin{bmatrix} -c & -d \\ a & b \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}b & \\ & \end{bmatrix}$$

$$A = \begin{matrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{matrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad \begin{bmatrix} \frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}b & -\frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}b \\ \frac{1}{\sqrt{2}}c + \frac{1}{\sqrt{2}}d & -\frac{1}{\sqrt{2}}c + \frac{1}{\sqrt{2}}d \end{bmatrix}$$

$\frac{1}{\sqrt{2}}(a+b) = -c$

$\frac{1}{\sqrt{2}}(-a+b) = -d$

$\frac{1}{\sqrt{2}}(c+d) = a$

T F

b. (2 points) If A and B are similar, and A is invertible, then B must be invertible as well.

$AS = SB$

$\cancel{AS}\ A =$

T F

c. (2 points) If A is an invertible matrix, then the equation $(A^{-1})^T = (A^T)^{-1}$ must hold.

T F

d. (2 points) For all 4×4 matrices A, $\det(4A) = 4\det A$.

16
16

T F

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \det A = 1 \quad \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 16^2$$

e. (2 points) If A is an $n \times n$ matrix such that $A = SBS^{-1}$, then $\det A^5 = \det B^5$.

$\det(A) = \det(B)$
if similar

T F

$$\begin{tikzcd} \vec{x} \arrow[r, "A"', "{\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}}"] & T(\vec{x}) \\ {[\vec{x}]_{\mathcal{B}}} \arrow[u, "S"] \arrow[r, "B"] & {[T(\vec{x})]_{\mathcal{B}}} \arrow[u, "S"'] \end{tikzcd}$$

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2. (10 points) Suppose that

$$T(\vec{x}) = A\vec{x} = \begin{pmatrix} \overset{v_1}{0} & \overset{v_2}{1} \\ -2 & 3 \end{pmatrix} \vec{x}$$

is a linear transformation expressed as the matrix A with respect to the standard basis $\mathcal{S} = (\vec{e}_1, \vec{e}_2)$. Let $\mathcal{B} = (\vec{v}_1, \vec{v}_2) = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right)$ be another basis for $\mathbb{R}^2$.

$c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

a. (7 points) Find the matrix $B = [T]_{\mathcal{B}}$, which expresses the linear transformation T as a matrix with respect to the basis $\mathcal{B}$.

$S = \cancel{\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}$ $\qquad B = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}\right)$ $\qquad AS = SB$

$[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 4 & -2 \end{array}\right] \xrightarrow{-(I)} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 2 & -2 \end{array}\right] \xrightarrow{\div 2} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{-2(II)} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 4 & 3 \end{array}\right] \xrightarrow{-(I)} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 2 & 2 \end{array}\right] \xrightarrow{\div 2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{-2(II)} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right]$$

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$$B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

b. (3 points) Find A^5. (Hint: your work in part 'a.' might be useful.)

$$\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -6 & 7 \\ -14 & 15 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -14 & 15 \\ -30 & 31 \end{bmatrix}$$

$$\begin{bmatrix} -14 & 15 \\ -30 & 31 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -30 & 31 \\ -62 & 63 \end{bmatrix}$$

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3. (10 points total) Let $A = \begin{pmatrix} 3 & 3 & 0 \\ 4 & 4 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{pmatrix}$.

a. (7 points) Find the QR-decomposition of A. Make sure to show your work.

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$\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 3 \\ 4 \\ 3 \\ 4 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix}$

$\vec{u}_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$

$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$

$= \begin{bmatrix} 3 \\ 4 \\ 3 \\ 4 \end{bmatrix} - \left(\frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 3 \\ 4 \end{bmatrix}\right)\frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 4 \\ 3 \\ 4 \end{bmatrix} - \left(\frac{1}{5}(25)\right)\left(\frac{1}{5}\right)\begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$

$\vec{u}_2 = \frac{1}{5}\begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$

$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_1$

$= \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix} - \left(\frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix}\right)\frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix} - (\vec{u}_3 \cdot \vec{v}_3)\vec{u}_2$

$\begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix} - \left(\frac{1}{5}\begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix}\right)\frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix}$

$\vec{u}_3 = \frac{1}{5}\begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 3 & 3 & 0 \\ 4 & 4 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 5 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$u_1 \cdot v_2$

$\frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 3 \\ 4 \end{bmatrix} = \frac{25}{5}$

$u_1 \cdot v_3 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix} =$

$u_2 \cdot v_3 = \frac{1}{5}\begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix} =$

b. (3 points) Find the matrix which represents orthogonal projection onto the subspace of $\mathbb{R}^4$ which is defined by the columns of A.

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QQ^T $\qquad Q^T = \frac{1}{5}\begin{bmatrix} 3 & 4 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & -4 & 3 \end{bmatrix}$

$\frac{1}{25}\begin{bmatrix} 3 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & -4 & 3 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 9 & 12 & 0 & 0 \\ 12 & 16 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 25 \end{bmatrix}$

4. (10 points) Find the equation of the line which best fits, in the least squares sense, the following points in the 'x, y'-plane: $(-1,-1), (0,1), (1,4)$.

$$c_0 + c_1 t \qquad (-1,-1) \quad (0,1) \quad (1,4)$$

$$\left| \begin{array}{ll} c_0 - c_1 & = -1 \\ c_0 & = 1 \\ c_0 + c_1 & = 4 \end{array} \right| \qquad A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_0^* \\ c_1^* \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

$$(A^TA)^{-1}A^T\vec{b}$$

$$\left[\begin{array}{cc|cc} 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} \div 3 \\ \div 2 \end{array} \qquad \left[\begin{array}{cc|cc} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 1/2 \end{array}\right]$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$(A^TA)^{-1}A^T = \frac{1}{6}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ -3 & 0 & 3 \end{bmatrix}$$

$$(A^TA)^{-1}A^T\vec{b} = \frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ -3 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 3/2 \end{bmatrix}$$

$$\boxed{y = \cancel{2+}\ \tfrac{3}{2}t} \qquad -1$$