Math 33A Exam # 2 Fall 2014

First and Last Name: ______________________ Initial of Last Name: ________

Student ID: ____________________ Discussion Section: ________________

Signature:

By signing here, you confirm you are the person identified above and that all the work herein is solely your own.

Instructions:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers. If you use the back of a page please indicate that you have work on the reverse side.
- You may not use books, notes, calculators, mobile phones, or any outside help during this midterm exam. You may not collaborate with other students in any way during this midterm exam.

Problem	Points	Score
1	10	4
2	10	4
3	10	10
4	10	8
5	10	2
Total	50	28

1. (10 points) The following problem consists of five statements about linear algebra. Indicate which of the following statements are true and which are false. For these problems you need not justify you answer. (Note that each correct answer is worth 2 points for the statements in problem 1).

(a) (2 points)
If the $n \times n$ matrix A is similar to the $n \times n$ matrix B, then $A^7 + 5I_n$ is similar to $B^7 + 5I_n$.
True or **False**

(b) (2 points)
For a subspace W of $\mathbb{R}^n$ and a vector $\vec{x}$ in $\mathbb{R}^n$ the length of the orthogonal projection of $\vec{x}$ onto W, $\text{proj}_W\vec{x}$, must be less than or equal to the length of $\vec{x}$. That is, $\|\text{proj}_W\vec{x}\| \leq \|\vec{x}\|$
True or **False**

(c) (2 points)
Every square matrix may be written uniquely as the sum of a symmetric matrix and a skew-symmetric matrix.
True or **False**

$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix}$

$a = x$ $w = c$

$d = \frac{z-y}{2}$

$b = \frac{z+y}{2}$

(d) (2 points)
For any $n \times m$ matrix A, $dim(im(A^T)) = dim(im(A))$.
True or **False**

$\text{rank}(A^T) = \text{rank}(A)$
$\|$
$\dim(\text{im})$

(e) (2 points)
If the matrix A is symmetric and the matrix B is orthogonal, then $B^{-1}AB$ is symmetric.
True or **False**

$\|$
B^T

2. (10 points)

Let $\mathcal{B} = (\vec{v_1}, \vec{v_2})$ be a basis for $\mathbb{R}^2$ where $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Let $\mathcal{C} = (\vec{w_1}, \vec{w_2})$ be another basis for $\mathbb{R}^2$ where $\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

As usual, let $\mathcal{S} = (\vec{e_1}, \vec{e_2})$ be the standard basis for $\mathbb{R}^2$.

(a) (2 points) What is the matrix which changes the $\mathcal{B}$ coordinates to the $\mathcal{S}$ coordinates?

$\mathcal{B} \to \mathcal{C}$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\vec{x}]_{\mathcal{C}}$$

$\mathcal{C} \to \mathcal{S}$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\vec{x}]_{\mathcal{S}}$$

$\mathcal{B} \to \mathcal{S}$

$$\cancel{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}} \quad \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

0

(b) (4 points) What is the matrix which changes the $\mathcal{S}$ coordinates to the $\mathcal{B}$ coordinates.

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$\mathcal{S} \to \mathcal{C}$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\vec{x}]_{\mathcal{S}}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\frac{1}{1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$\mathcal{C} \to \mathcal{B}$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$\frac{1}{1}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\mathcal{S} \to \mathcal{B}$

$$\cancel{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}} \quad \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

(c) (4 points) If $[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is the $\mathcal{C}$ coordinate vector for $\vec{x}$, find $[\vec{x}]_{\mathcal{B}}$, the $\mathcal{B}$ coordinate vector for $\vec{x}$.

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

0

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 4 \end{array}\right]$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

3. (10 points)

(a) (7 points) Find the QR-decomposition of the following matrix, whose columns are linearly independent:

$$A = \begin{array}{c} \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \\ \begin{bmatrix} 0 & -1 & -2 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \end{array}$$

$A = QR$ $\quad Q = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

$\vec{u}_1 = \frac{\vec{v}_1}{r_{11}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\qquad r_{11} = \|\vec{v}_1\| = 2$

$\vec{v}_2^{\perp} = \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_2} - \underbrace{1}_{r_{12}} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}_1} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ $\qquad r_{12} = \vec{u}_1 \cdot \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 1$

$\qquad r_{22} = \|\vec{v}_2^{\perp}\| = 1$

$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{r_{22}} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ $\qquad r_{13} = \vec{u}_1 \cdot \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = 1$

$\vec{v}_3^{\perp} = \underbrace{\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}}_{\vec{v}_3} - \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{r_{13}\vec{u}_1} + \underbrace{\begin{bmatrix} +2 \\ 0 \\ 0 \end{bmatrix}}_{r_{23}\vec{u}_2} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ $\qquad r_{23} = \vec{u}_2 \cdot \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} = 2$

$\qquad r_{33} = \|\vec{v}_3^{\perp}\| = 2$

$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{r_{33}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

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(b) (3 points) For the matrix Q that you obtained in part (a) find Q^{-1}.

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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4. (10 points)
Use the method of least squares approximation to find the equation of the line which best fits the following points in the xy-plane: $(-2,-3), (-1,-1), (0,1), (2,2)$. When you've found the line, sketch its graph in the xy-plane along with the four original data points.

$f(x) = c_1 x + c_0$

$f(-2) = -2c_1 + c_0 = -3$

$-c_1 + c_0 = -1$

$c_0 = 1$

$2c_1 + c_0 = 2$

$$\underbrace{\begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}}_{A,\ 4\times 2} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\vec{x},\ 2\times 1} = \underbrace{\begin{bmatrix} -3 \\ -1 \\ 1 \\ 2 \end{bmatrix}}_{\vec{b},\ 4\times 1}$$

4

Approximation: Least squares

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$$

4

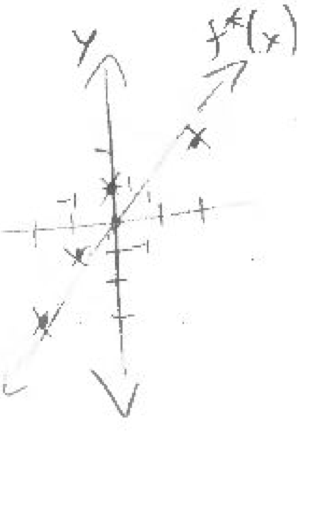
$$= \left(\begin{bmatrix} -2 & -1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}\right)^{-1} A^T\vec{b}$$

$$= \left(\begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix}\right)^{-1} A^T\vec{b}$$

$$= \frac{1}{35}\underbrace{\begin{bmatrix} 4 & 1 \\ 1 & 9 \end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix} -2 & -1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{2\times 4}\vec{b}$$

$$= \frac{1}{37}\underbrace{\begin{bmatrix} -7 & -3 & 1 & 9 \\ 7 & 8 & 9 & 11 \end{bmatrix}}_{2\times 4}\underbrace{\begin{bmatrix} -3 \\ -1 \\ 1 \\ 2 \end{bmatrix}}_{4\times 1}$$

$$\vec{x} = \frac{1}{37}\begin{bmatrix} 25+18 \\ -29+9+22 \end{bmatrix} = \frac{1}{37}\begin{bmatrix} 43 \\ 2 \end{bmatrix}$$



$$\boxed{f(x) \approx \tfrac{43}{37}x + \tfrac{2}{37}}$$

5. (10 points)

For the equation $A\vec{x} = \vec{b}$, calculate the least-squares solution, $\vec{x}^*$, which has minimal length, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

and

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$A\vec{x} = \vec{b}$

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

$$= \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \right)^{-1} A^T \vec{b}$$

$$= \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix}^{-1} A^T \vec{b}$$

$=$

Not invertible

So use:

$$A^T A \vec{x}^* = A^T \vec{b}$$

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