1B

Mathematics 33A Exam #1 Fall Quarter 2016

Last Name:________ First Name:________

Student ID number

Instructions:

- You have 50 minutes for this exam.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers. If you use the back of a page please indicate that you have work on the reverse side.
- You may not use books, notes, calculators, mobile phones, or any outside help during this exam. You may not collaborate with other students in any way during this exam.

Problem	Points	Score
1	10	6
2	10	10
3	10	8
4	10	~~7~~ 8
Total	40	32

6

1. (10 points total) Circle "T" if the statement is true and "F" if false. You need not show your work. No partial credit will be given.

   a. (2 points) If A and B are matrices such that $C = AB$ exists and C^{-1} exists, then A and B are both invertible.

      T F

      $(AB)^{-1} = A^{-1}B^{-1}$

   b. (2 points) If $AB = 0$ then it is always true that either $A = 0$ or $B = 0$. Here 0 is the zero matrix.

      T F

   c. (2 points) If A is an $n \times n$ matrix such that $A^2 = A$, and A is not the identity matrix I_n, then $\ker(I_n - A) = \mathrm{im}(A)$.

      T F

   d. (2 points) There is a square matrix whose image equals its kernel.

      T F

   e. (2 points) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x \\ y-1 \end{pmatrix}$ is a linear transformation.

      T F

      $T(x+y) = T(x) + T(y)$

10

2. (10 points total) The following problem has parts a) and b):

a. (5 points) Find the reduced row-echelon form of

$$\begin{pmatrix} 1 & 6 & 7 & 1 & 0 & 0 \\ 1 & 6 & 6 & 0 & 1 & 0 \\ 1 & 5 & 4 & 0 & 0 & 1 \end{pmatrix} = \begin{vmatrix} 1 & 6 & 7 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 7 & 1 & 0 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & -11 & -5 & 0 & 6 \\ 0 & -1 & -3 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & -11 & -5 & 0 & 6 \\ 0 & 1 & 3 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{vmatrix} = \boxed{\begin{vmatrix} 1 & 0 & 0 & 6 & -11 & 6 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{vmatrix} = \text{rref}}$$

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rref(A):
- first nonzero entry in row is leading 1
- If leading 1 in column, all other entries are zero
- If leading 1, then there is leading 1 in row above, and to the left

b. (5 points) Solve the equation $A\vec{x} = \vec{b}$ for $\vec{x}$ when

$$A = \begin{pmatrix} 6 & -11 & 6 \\ -2 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$$

$$A^{-1} = \begin{vmatrix} 1 & 0 & 0 & 1 & 6 & 7 \\ 0 & 1 & 0 & 1 & 6 & 6 \\ 0 & 0 & 1 & 1 & 5 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 6 & -11 & 6 & 1 & 0 & 0 \\ -2 & 3 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 6 & -11 & 6 & 1 & 0 & 0 \\ 0 & -2 & 3 & 1 & 3 & 0 \\ 0 & 5 & -6 & -1 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 12 & 0 & -21 & -9 & -33 & 0 \\ 0 & -2 & 3 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 12 & 0 & 0 & 12 & 72 & 84 \\ 0 & 2 & 0 & 2 & 12 & 12 \\ 0 & 0 & 1 & 1 & 5 & 4 \end{vmatrix}$$

$$\vec{x} = \begin{vmatrix} 1 & 6 & 7 \\ 1 & 6 & 6 \\ 1 & 5 & 4 \end{vmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(1)+6(1)+7(1) \\ 1(1)+6(1)+6(1) \\ 1(1)+5(1)+4(1) \end{bmatrix} = \boxed{\begin{bmatrix} 14 \\ 13 \\ 10 \end{bmatrix}}$$

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3. (10 points total) Let $V = span(v_1, v_2, v_3, v_4, v_5)$, where

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix}, v_5 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}.$$

a. (2 points) Are the vectors whose span defines V linearly independent or linearly dependent? Explain why or why not.

• Linearly independent if no vectors are redundant (linear combinations) of the other vectors

The vectors are linear dependent because $\vec{v_2}$ is redundant of $\vec{v_1}$ & $\vec{v_3}$. It take just one redundant vector for all vectors to be $\boxed{\text{linear dependent}}$. Also, $\vec{v_5}$ is redundant of $\vec{v_2}$ & $\vec{v_3}$

b. (4 points) Find a basis for V.

• Basis is the span of all linearly independent vectors in V.

• $\vec{v_2}$ and $\vec{v_5}$ are redundant, and will be omitted

$$\text{basis of } V = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}\right)$$

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c. (4 points) Find the kernel of the matrix $A = [v_1\ v_2\ v_3\ v_4\ v_5]$.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 3 & 0 \\ 2 & 3 & 1 & 7 & 4 & 0 \\ 3 & 4 & 1 & 11 & 5 & 0 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & -1 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 5 & -1 & 0 \\ 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = x_3 - 5x_4 + x_5$

$x_2 = -x_3 + x_4 - 2x_5$

$x_3 = r,\ x_4 = s,\ x_5 = t$

$$\begin{array}{r} 3\ 4\ 1\ 11\ 5 \\ -\ 3\ 6\ 3\ 9\ 9 \\ \hline 0\ -2\ -2\ 2\ -4 \end{array} \qquad \begin{array}{r} 2\ 3\ 1\ 7\ 4 \\ -\ 2\ 4\ 2\ 6\ 6 \\ \hline 0\ -1\ -1\ 1\ -2 \end{array} \qquad \begin{array}{r} 1\ 2\ 1\ 3\ 3 \\ -\ 0\ 2\ 2\ -2\ 4 \\ \hline 1\ 0\ -1\ 5\ -1 \end{array} \qquad \begin{array}{r} 0\ 1\ 1\ -1\ 2 \\ -\ 0\ 1\ 1\ -1\ 2 \\ \hline \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} r - 5s + t \\ -r + s - 2t \\ r \\ s \\ t \end{bmatrix}$$

$$\boxed{\ker(A) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)}$$

4. (10 points total) The following problem has parts a) and b):

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a. (6 points) Find the 3×3 matrix which represents orthogonal projection onto the line spanned by $v = \begin{pmatrix}1\\3\\4\end{pmatrix}$.

$$\mathrm{proj}_L(\vec{x}) = \left(\frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \left(\frac{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\cdot\begin{bmatrix}1\\3\\4\end{bmatrix}}{\begin{bmatrix}1\\3\\4\end{bmatrix}\cdot\begin{bmatrix}1\\3\\4\end{bmatrix}}\right)\begin{bmatrix}1\\3\\4\end{bmatrix} = \left(\frac{x_1+3x_2+4x_3}{1+9+16}\right)\begin{bmatrix}1\\3\\4\end{bmatrix}$$

$$= \left(\frac{x_1+3x_2+4x_3}{26}\right)\begin{bmatrix}1\\3\\4\end{bmatrix}$$

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{26}\begin{bmatrix}1\\3\\4\end{bmatrix},\quad T\begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{3}{26}\begin{bmatrix}1\\3\\4\end{bmatrix},\quad T\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{4}{26}\begin{bmatrix}1\\3\\4\end{bmatrix} =$$

$$\boxed{A = \frac{1}{26}\begin{bmatrix}1&3&4\\3&9&12\\4&12&16\end{bmatrix}}$$

b. (4 points) Find the matrix which represents reflection across the plane in $\mathbb{R}^3$ defined by the equation $x + 3y + 4z=0$.

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$\ker(A) = \begin{bmatrix}1\\3\\4\end{bmatrix}$ vector

$\mathrm{im}(A) = $ plane that is $x+3y+4z=0$

$$\mathrm{ref}_L(\vec{x}) = 2\,\mathrm{proj}_L(\vec{x}) - \vec{x}$$

$$\mathrm{ref}_V(\vec{x}) = \vec{x} - 2\,\mathrm{proj}_L(\vec{x})$$

$$2\,\mathrm{proj}_L(\vec{x}) = 2\left(\frac{1}{26}\begin{bmatrix}1&3&4\\3&9&12\\4&12&16\end{bmatrix}\right) = \frac{1}{13}\begin{bmatrix}1&3&4\\3&9&12\\4&12&16\end{bmatrix}$$

$$\mathrm{ref}_V(\vec{x}) = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} - \frac{1}{13}\begin{bmatrix}1&3&4\\3&9&12\\4&12&16\end{bmatrix} = \boxed{\begin{bmatrix}x_1-1/13 & x_1-3/13 & x_1-4/13\\x_2-3/13 & x_2-9/13 & x_2-12/13\\x_3-4/13 & x_3-12/13 & x_3-16/13\end{bmatrix}}$$