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Mathematics 33A

Exam #1

Fall Quarter 2016

Last Name: _____

First Name: _____

Student ID number _____

Instructions:

- You have 50 minutes for this exam.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers. If you use the back of a page please indicate that you have work on the reverse side.
- You may not use books, notes, calculators, mobile phones, or any outside help during this exam. You may not collaborate with other students in any way during this exam.

Problem	Points	Score
1	10	6
2	10	10
3	10	8
4	10	8
Total	40	32

1. (10 points total) Circle "T" if the statement is true and "F" if false. You need not show your work. No partial credit will be given.

a. (2 points) If A and B are matrices such that $C = AB$ exists and C^{-1} exists, then A and B are both invertible.

~~T~~ F

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$C^{-1} =$$

$$AB = I \implies BA = I \implies A = B^{-1}, B = A^{-1}$$

b. (2 points) If $AB = 0$ then it is always true that either $A = 0$ or $B = 0$. Here 0 is the zero matrix.

T ~~F~~

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1(0)+1(0) & 1(1)+1(-1) \\ 1(0)+1(0) & 1(1)+1(-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix}$$

c. (2 points) If A is an $n \times n$ matrix such that $A^2 = A$, and A is not the identity matrix I_n , then $\ker(I_n - A) = \text{im}(A)$.

~~T~~ F

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1(1)+0(0) & 1(0)+0(0) \\ 0(1)+0(0) & 0(0)+0(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\ker(I_2 - A) = \text{im}(A)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad x_1 = x_2, x_2 = r$$

$$\ker(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\text{im}(A) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

d. (2 points) There is a square matrix whose image equals its kernel.

T ~~F~~

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \ker(A) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{im}(A) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \ker(A) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \text{im}(A) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}, \ker(A) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \text{im}(A) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$$

$$x_1 = 0, x_2 = r \implies \begin{bmatrix} 0 \\ r \end{bmatrix}$$

$$x_1 = -x_2$$

e. (2 points) $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x \\ y-1 \end{pmatrix}$ is a linear transformation.

T ~~F~~

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ x \\ y-1 \end{bmatrix}$$

$$T(x+y) = T(x) + T(y)$$

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2. (10 points total) The following problem has parts a) and b):

a. (5 points) Find the reduced row-echelon form of

$$\begin{pmatrix} 1 & 6 & 7 & 1 & 0 & 0 \\ 1 & 6 & 6 & 0 & 1 & 0 \\ 1 & 5 & 4 & 0 & 0 & 1 \end{pmatrix} = \left[\begin{array}{cccccc|ccc} 1 & 6 & 7 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 & 0 & 0 & 0 \end{array} \right] = \left[\begin{array}{cccccc|ccc} 1 & 6 & 7 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{array}{r} \begin{array}{cccccc|ccc} 1 & 6 & 6 & 0 & 1 & 0 & 0 & 0 & 0 \\ - & 1 & 6 & 7 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \end{array} & + & \begin{array}{cccccc|ccc} 1 & 5 & 4 & 0 & 0 & 1 & 0 & 0 & 0 \\ - & 1 & 6 & 7 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & -3 & -1 & 0 & 1 & 0 & 0 & 0 \end{array} & + & \begin{array}{cccccc|ccc} 1 & 6 & 7 & 1 & 0 & 0 & 0 & 0 & 0 \\ - & 0 & -6 & -10 & -6 & 0 & 6 & 0 & 0 \\ \hline 1 & 0 & -11 & -5 & 0 & 6 & 0 & 0 & 0 \end{array} & + & \begin{array}{cccccc|ccc} 1 & 0 & -11 & -5 & 0 & 6 & 0 & 0 & 0 \\ - & 0 & -1 & -3 & -1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \end{array} \end{array}$$

$$\left[\begin{array}{cccccc|ccc} 1 & 0 & -11 & -5 & 0 & 6 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \end{array} \right] = \left[\begin{array}{cccccc|ccc} 1 & 0 & 0 & 6 & -11 & 6 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \end{array} \right] = \text{rref}$$

$$\begin{array}{r} \begin{array}{cccccc|ccc} 1 & 0 & -11 & -5 & 0 & 6 & 0 & 0 & 0 \\ - & 0 & 0 & 11 & 11 & -11 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 6 & -11 & 6 & 0 & 0 & 0 \end{array} & + & \begin{array}{cccccc|ccc} 0 & 1 & 3 & 1 & 0 & -1 & 0 & 0 & 0 \\ - & 0 & 0 & 3 & 3 & -3 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & -2 & 3 & -1 & 0 & 0 & 0 \end{array} \end{array}$$

rref(A):

- first nonzero entry in row is leading 1
- If leading 1 in column, all other entries are zero
- If leading 1, then there is leading 1 in row above, and to the left

b. (5 points) Solve the equation $A\vec{x} = \vec{b}$ for \vec{x} when

$$A = \begin{pmatrix} 6 & -11 & 6 \\ -2 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 6 & 7 \\ 0 & 1 & 0 & 1 & 6 & 6 \\ 0 & 0 & 1 & 1 & 5 & 4 \end{array} \right]$$

$$A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$$

$$\left[\begin{array}{ccc|ccc} 6 & -11 & 6 & 1 & 0 & 0 \\ -2 & 3 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right] = \left[\begin{array}{ccc|ccc} 6 & -11 & 6 & 1 & 0 & 0 \\ 0 & -2 & 3 & 1 & 1 & 0 \\ 0 & 5 & -6 & -1 & 0 & 6 \end{array} \right] = \left[\begin{array}{ccc|ccc} 12 & 0 & -21 & -9 & -33 & 0 \\ 0 & -2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 5 & 4 \end{array} \right] = \left[\begin{array}{ccc|ccc} 12 & 0 & 0 & 12 & 72 & 84 \\ 0 & 2 & 0 & 2 & 12 & 12 \\ 0 & 0 & 1 & 1 & 5 & 4 \end{array} \right]$$

$$\begin{array}{r} \begin{array}{ccc|ccc} -6 & 9 & -3 & 0 & 3 & 0 \\ 6 & -11 & 6 & 1 & 0 & 0 \\ \hline 0 & -2 & 3 & 1 & 3 & 0 \end{array} & + & \begin{array}{ccc|ccc} 6 & -6 & 0 & 0 & 0 & 6 \\ -6 & -11 & 6 & 1 & 0 & 0 \\ \hline 0 & 5 & -6 & -1 & 0 & 6 \end{array} & - & \begin{array}{ccc|ccc} 12 & -22 & 12 & 2 & 0 & 0 \\ 0 & -22 & 33 & 11 & 33 & 0 \\ \hline 12 & 0 & -21 & -9 & -33 & 0 \end{array} & + & \begin{array}{ccc|ccc} 0 & 10 & -12 & -2 & 0 & 12 \\ 0 & -10 & 15 & 5 & 15 & 0 \\ \hline 0 & 0 & 3 & 3 & 15 & 12 \end{array} \end{array}$$

$$\begin{array}{r} \begin{array}{ccc|ccc} 12 & 0 & -21 & -9 & -33 & 0 \\ 0 & 0 & 21 & 21 & 105 & 84 \\ \hline 12 & 0 & 0 & 12 & 72 & 84 \end{array} & - & \begin{array}{ccc|ccc} 0 & -2 & 3 & 1 & 3 & 0 \\ 0 & -2 & 3 & 1 & 3 & 0 \\ \hline 0 & -2 & 0 & -2 & -12 & -12 \end{array} & \Rightarrow & \vec{x} = \begin{bmatrix} 1 & 6 & 7 \\ 1 & 6 & 6 \\ 1 & 5 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(1) + 6(1) + 7(1) \\ 1(1) + 6(1) + 6(1) \\ 1(1) + 5(1) + 4(1) \end{bmatrix} \\ & & & & & = \begin{bmatrix} 14 \\ 13 \\ 10 \end{bmatrix} \end{array}$$

3. (10 points total) Let $V = \text{span}(v_1, v_2, v_3, v_4, v_5)$, where

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix}, v_5 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}.$$

a. (2 points) Are the vectors whose span defines V linearly independent or linearly dependent? Explain why or why not.

• Linearly independent if no vectors are redundant (linear combinations) of the other vectors

The vectors are linear dependent because v_2 is redundant of v_1, v_3 . If take just one redundant vector for all vectors to be linear dependent. Also, v_5 is redundant of v_2, v_3 .

b. (4 points) Find a basis for V .

• Basis is the span of all linearly independent vectors in V .

• v_2 and v_5 are redundant, and will be omitted

$$\text{basis of } V = \text{span} \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix} \right)$$

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c. (4 points) Find the kernel of the matrix $A = [v_1 \ v_2 \ v_3 \ v_4 \ v_5]$.

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 3 & | & 0 \\ 2 & 3 & 1 & 7 & 4 & | & 0 \\ 3 & 4 & 1 & 11 & 5 & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 3 & 3 & | & 0 \\ 0 & 1 & -1 & 4 & 1 & | & 0 \\ 0 & 1 & -1 & 8 & 2 & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 5 & -1 & | & 0 \\ 0 & 1 & -1 & 4 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 = x_3 - 5x_4 + x_5$$

$$x_2 = -x_3 + x_4 - 2x_5$$

$$x_3 = r, x_4 = s, x_5 = t$$

$$\begin{array}{r} 3 \ 4 \ 1 \ 11 \ 5 \\ -2 \ 3 \ 1 \ 7 \ 4 \\ \hline 3 \ 6 \ 3 \ 9 \ 9 \\ \hline 0 \ -2 \ -2 \ 2 \ -4 \end{array}$$

$$\begin{array}{r} 2 \ 3 \ 1 \ 7 \ 4 \\ -2 \ 4 \ 2 \ 6 \ 6 \\ \hline 0 \ -1 \ -1 \ 1 \ -2 \end{array}$$

$$\begin{array}{r} 1 \ 2 \ 1 \ 3 \ 3 \\ -0 \ 2 \ 2 \ 2 \ 4 \\ \hline 1 \ 0 \ -1 \ 5 \ -1 \end{array} \quad \begin{array}{r} 0 \ 1 \ 1 \ -1 \ 2 \\ 0 \ 1 \ 1 \ -1 \ 2 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} r - 5s + t \\ -r + s - 2t \\ r \\ s \\ t \end{bmatrix}$$

$$\text{ker}(A) = \text{span} \left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

4. (10 points total) The following problem has parts a) and b):

6 a. (6 points) Find the 3×3 matrix which represents orthogonal

projection onto the line spanned by $v = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$.

$$\text{proj}_L(\vec{x}) = \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}}{\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \frac{x_1 + 3x_2 + 4x_3}{1+9+16} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \frac{x_1 + 3x_2 + 4x_3}{26} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{26} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{3}{26} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{4}{26} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} =$$

$$A = \frac{1}{26} \begin{bmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ 4 & 12 & 16 \end{bmatrix}$$

b. (4 points) Find the matrix which represents reflection across the plane in \mathbb{R}^3 defined by the equation $x + 3y + 4z = 0$.

ker(A) = $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ vector
in(A) = plane that is $x + 3y + 4z = 0$

$$\text{ref}_L(\vec{x}) = 2 \text{proj}_L(\vec{x}) - \vec{x}$$

$$\text{ref}_V(\vec{x}) = \vec{x} - 2 \text{proj}_L(\vec{x})$$

$$2 \text{proj}_L(\vec{x}) = 2 \left(\frac{1}{26} \begin{bmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ 4 & 12 & 16 \end{bmatrix} \vec{x} \right) = \frac{1}{13} \begin{bmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ 4 & 12 & 16 \end{bmatrix} \vec{x}$$

$$\text{ref}_V(\vec{x}) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \frac{1}{13} \begin{bmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ 4 & 12 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 1/13 & x_1 - 3/13 & x_1 - 4/13 \\ x_2 - 3/13 & x_2 - 9/13 & x_2 - 12/13 \\ x_3 - 4/13 & x_3 - 12/13 & x_3 - 16/13 \end{bmatrix}$$