2

1. (10 points total) Circle "T" if the statement is true and "F" if false. You need not show your work. No partial credit will be given.

   a. (2 points) If A and B are matrices such that $C = AB$ exists and C^{-1} exists, then A and B are both invertible.

   $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =$

   invertible, not invertible

   T F

   b. (2 points) If $AB = 0$ then it is always true that either $A = 0$ or $B = 0$. Here 0 is the zero matrix.

   $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix} = 0$

   T F

   c. (2 points) If A is an $n \times n$ matrix such that $A^2 = A$, and A is not the identity matrix I_n, then $\ker(I_n - A) = \mathrm{im(A)}$.

   $\mathrm{Im}(A)$

   $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

   $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

   $I_n - A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

   T F

   $\mathrm{Im}(A) = \vec{a}_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \vec{a}_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

   d. (2 points) There is a square matrix whose image equals its kernel.

   $\mathrm{im} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + $

   T F

   $\ker \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$

   e. (2 points) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x \\ y - 1 \end{pmatrix}$ is a linear transformation.

   $x \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

   T F

10

2. (10 points total) The following problem has parts a) and b):

a. (5 points) Find the reduced row-echelon form of

$$\begin{pmatrix}1&6&7&1&0&0\\1&6&6&0&1&0\\1&5&4&0&0&1\end{pmatrix}\begin{matrix}\\ -I\\ -I\end{matrix} \Rightarrow \begin{pmatrix}1&6&7&1&0&0\\0&0&-1&-1&1&0\\0&-1&-3&-1&0&1\end{pmatrix} +6(III)$$

$$\begin{pmatrix}1&0&-11&-5&0&6\\0&0&-1&-1&1&0\\0&-1&-3&-1&0&1\end{pmatrix}\begin{matrix}-(II)(11)\\ \\ -(II)(3)\end{matrix} = \begin{pmatrix}1&0&0&6&-11&6\\0&0&-1&-1&1&0\\0&-1&0&2&-3&1\end{pmatrix}\begin{matrix}\\ (\cdot -1)\\ (\cdot -1)\end{matrix} \Rightarrow \begin{pmatrix}1&0&0&6&-11&6\\0&0&1&1&-1&0\\0&1&0&-2&3&-1\end{pmatrix}$$

$$\boxed{\begin{pmatrix}1&0&0&6&-11&6\\0&1&0&-2&3&-1\\0&0&1&1&-1&0\end{pmatrix}}$$

✓

b. (5 points) Solve the equation $A\vec{x} = \vec{b}$ for $\vec{x}$ when

$$A = \begin{pmatrix}6&-11&6\\-2&3&-1\\1&-1&0\end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

$\vec{x} = A^{-1}b$

inverse of A $\rightarrow$

$$\left(\begin{array}{ccc|ccc}6&-11&6&1&0&0\\-2&3&-1&0&1&0\\1&-1&0&0&0&1\end{array}\right) -(5)III \Rightarrow \left(\begin{array}{ccc|ccc}1&-6&6&1&0&-5\\-2&3&-1&0&1&0\\1&-1&0&0&0&1\end{array}\right)\begin{matrix}\\ +2(I)\\ -(I)\end{matrix}$$

$$\left(\begin{array}{ccc|ccc}1&-6&6&1&0&-5\\0&-9&11&2&1&-10\\0&5&-6&-1&0&6\end{array}\right) +(2)(III) = \left(\begin{array}{ccc|ccc}1&-6&6&1&0&-5\\0&1&-1&0&1&2\\0&5&-6&-1&0&6\end{array}\right)\begin{matrix}+II(6)\\ \\ -(II)(5)\end{matrix} \Rightarrow \left(\begin{array}{ccc|ccc}1&0&0&1&6&7\\0&1&-1&0&1&2\\0&0&-1&-1&-5&-4\end{array}\right) -(III)$$

$$\left(\begin{array}{ccc|ccc}1&0&0&1&6&7\\0&1&0&1&6&6\\0&0&-1&-1&-5&-4\end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc}1&0&0&1&6&7\\0&1&0&1&6&6\\0&0&1&1&5&4\end{array}\right)$$

✓

$$\vec{x} = \begin{pmatrix}1&6&7\\1&6&6\\1&5&4\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}14\\13\\10\end{pmatrix}$$

3. (10 points total) Let $V = span(v_1, v_2, v_3, v_4, v_5)$, where

$$v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}, v_2 = \begin{pmatrix}2\\3\\4\end{pmatrix}, v_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}, v_4 = \begin{pmatrix}3\\7\\11\end{pmatrix}, v_5 = \begin{pmatrix}3\\4\\5\end{pmatrix}.$$

a. (2 points) Are the vectors whose span defines V linearly independent or linearly dependent? Explain why or why not.

their are not all independent as V_1 can be combined linearly with V_3 to make V_2 and V_5. this mean that overall, there's a linear dependency.

how? -1

b. (4 points) Find a basis for V.

basis = (V_1, V_3, V_4) → as each is linearly independent.

show why? -3

The basis must be made of linearly independent vectors

c. (4 points) Find the kernel of the matrix $A = [v_1\ v_2\ v_3\ v_4\ v_5]$.

$$\begin{pmatrix}1&2&1&3&3\\2&3&1&7&4\\3&4&1&11&5\end{pmatrix}\begin{matrix}\\-2(I)\\-3(I)\end{matrix} \Rightarrow \begin{pmatrix}1&2&1&3&3\\0&-1&-1&1&-2\\0&-2&-1&2&-4\end{pmatrix}\begin{matrix}+(II)(2)\\ \\-(II)(2)\end{matrix} \Rightarrow \begin{pmatrix}1&0&1&5&-1\\0&-1&-1&1&-2\\0&0&1&0&0\end{pmatrix}\begin{matrix}-(III)\\+(III)\\ \end{matrix} = \begin{pmatrix}1&0&0&5&-1\\0&-1&0&1&-2\\0&0&1&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&5&1\\0&1&0&-1&2\\0&0&1&0&0\end{pmatrix}$$

$x_1 + 5x_4 + x_5 = 0$

$x_2 - x_4 + 2x_5 = 0$

$x_3 = 0$

$x_1 = -5x_4 - x_5$

$x_2 = x_4 - 2x_5$

$x_3 = 0$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}-5t-s\\t-2s\\0\\t\\s\end{pmatrix} = t\begin{pmatrix}-5\\1\\0\\1\\0\end{pmatrix} + s\begin{pmatrix}-1\\-2\\0\\0\\1\end{pmatrix}$$

kernel = span $\left(\begin{pmatrix}-5\\1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\-2\\0\\0\\1\end{pmatrix}\right)$

-2

4. (10 points total) The following problem has parts a) and b):

$V \cdot \frac{1}{\sqrt{26}}\begin{pmatrix}1\\3\\4\end{pmatrix} \frac{1}{\sqrt{26}}\begin{pmatrix}1\\3\\4\end{pmatrix}$

a. (6 points) Find the 3×3 matrix which represents orthogonal projection onto the line spanned by $v = \begin{pmatrix}1\\3\\4\end{pmatrix}$.

$\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}$

$\frac{1}{1+9+16}$ $\quad \mathrm{Proj}_V(x) = \frac{\vec{x}\cdot\vec{v}}{|v|^2}\vec{v}$ $\quad \frac{1}{26}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1\\3\\4\end{pmatrix}$

$|v|^2 = 26$

$\frac{1}{26}\begin{pmatrix}1\\3\\4\end{pmatrix}, \frac{3}{26}\begin{pmatrix}1\\3\\4\end{pmatrix}, \frac{4}{26}\begin{pmatrix}1\\3\\4\end{pmatrix}$

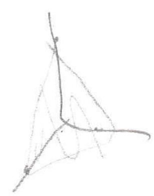
$= \frac{1}{26}\begin{pmatrix}1&3&4\\3&9&12\\4&12&16\end{pmatrix}$

$\begin{pmatrix}\frac{1}{26}&0&0\\0&3/26&0\\0&0&4/26\end{pmatrix}$

X

b. (4 points) Find the matrix which represents reflection across the plane in $\mathbb{R}^3$ defined by the equation $x + 3y + 4z = 0$.

$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1\\3\\4\end{pmatrix} = \begin{pmatrix}1&0&0\\0&3&0\\0&0&4\end{pmatrix}$



$\mathrm{reflection}_L \vec{x} = 2\,\mathrm{Proj}_L \vec{x} - \vec{x}$

X

$\approx \frac{1}{13}\begin{pmatrix}1&3&4\\3&9&12\\4&12&16\end{pmatrix} - ?$