Mathematics 33A Exam #1 Fall Quarter 2016

Last Name:______________________ First Name:______________________

Student ID number:_______________

Instructions:

- You have 50 minutes for this exam.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers. If you use the back of a page please indicate that you have work on the reverse side.
- You may not use books, notes, calculators, mobile phones, or any outside help during this exam. You may not collaborate with other students in any way during this exam.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
Total	40	

1. (10 points total) Circle "T" if the statement is true and "F" if false. You need not show your work. No partial credit will be given.

   a. (2 points) If A and B are matrices such that $C = AB$ exists and C^{-1} exists, then A and B are both invertible.

   T F

   If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

   $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = C$ & C^{-1} exists, but A & B aren't invertible

   b. (2 points) If $AB = 0$ then it is always true that either $A = 0$ or $B = 0$. Here 0 is the zero matrix.

   T F

   If $A = [1\ 0]$ & $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then

   $AB = 0$, but neither A nor B is zero

   c. (2 points) If A is an $n \times n$ matrix such that $A^2 = A$, and A is not the identity matrix I_n, then $\ker(I_n - A) = \mathrm{im}(A)$.

   T F

   $(I_n - A)A\vec{x} = (A - A^2)\vec{x} = (A - A)\vec{x} = 0\vec{x} = \vec{0}$

   $\Rightarrow A\vec{x} \in \ker(I_n - A)$. Hence $\mathrm{im}(A) \subset \ker(I_n - A)$

   Now if $(I_n - A)\vec{x} = 0$, $\vec{x} = A\vec{x} \Rightarrow \ker(I_n - A) \subset \mathrm{im}(A)$

   $\therefore \ker(I_n - A) = \mathrm{im}(A)$

   d. (2 points) There is a square matrix whose image equals its kernel.

   T F

   $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}$ $\Rightarrow \mathrm{im}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

   $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

   $\Rightarrow \ker\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

   e. (2 points) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x \\ y - 1 \end{pmatrix}$ is a linear transformation.

   T F

   $T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \Rightarrow T$ is

   not linear, since $T(\vec{0}) = \vec{0}$ for all linear transformations

2. (10 points total) The following problem has parts a) and b):

a. (5 points) Find the reduced row-echelon form of

$$M := \begin{pmatrix} 1 & 6 & 7 & 1 & 0 & 0 \\ 1 & 6 & 6 & 0 & 1 & 0 \\ 1 & 5 & 4 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} r_2 \to r_2 - r_1 \\ r_3 \to r_3 - r_1 \end{array}$$

$$\to \begin{pmatrix} 1 & 6 & 7 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{pmatrix} \xrightarrow[r_2 \,\&\, r_3]{\text{swap}} \begin{pmatrix} 1 & 6 & 7 & 1 & 0 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{pmatrix} \begin{array}{l} r_2 \to -r_2 \\ r_3 \to -r_3 \end{array}$$

$$\to \begin{pmatrix} 1 & 6 & 7 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{pmatrix} \begin{array}{l} r_1 \to r_1 - 7r_3 \\ r_2 \to r_2 - 3r_3 \end{array} \rightsquigarrow \begin{pmatrix} 1 & 6 & 0 & -6 & 7 & 0 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{pmatrix} r_1 \mapsto r_1 - 6r_2$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 6 & 11 & 6 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{pmatrix} = \mathrm{rref}(M)$$

Note that the process of computing $\mathrm{rref}(M)$ gives $A = \begin{pmatrix} 1 & 6 & 7 \\ 1 & 6 & 6 \\ 1 & 5 & 4 \end{pmatrix}^{-1} \iff A^{-1} = \begin{pmatrix} 1 & 6 & 7 \\ 1 & 6 & 6 \\ 1 & 5 & 4 \end{pmatrix}$

b. (5 points) Solve the equation $A\vec{x} = \vec{b}$ for $\vec{x}$ when

$$A = \begin{pmatrix} 6 & -11 & 6 \\ -2 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus $\vec{x} = A^{-1}\vec{b} = \begin{pmatrix} 1 & 6 & 7 \\ 1 & 6 & 6 \\ 1 & 5 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 10 \end{pmatrix}$

$\therefore \vec{x} = \begin{pmatrix} 14 \\ 13 \\ 10 \end{pmatrix}$

3. (10 points total) Let $V = span(v_1, v_2, v_3, v_4, v_5)$, where

$$v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}, v_2 = \begin{pmatrix}2\\3\\4\end{pmatrix}, v_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}, v_4 = \begin{pmatrix}3\\7\\11\end{pmatrix}, v_5 = \begin{pmatrix}3\\4\\5\end{pmatrix}.$$

a. (2 points) Are the vectors whose span defines V linearly independent or linearly dependent? Explain why or why not.

$V_2 = \begin{pmatrix}2\\3\\4\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix} = V_1 + V_3$ so the vectors $V_1, \dots, V_5$ whose span define V are linearly dependent

b. (4 points) Find a basis for V.

Let $A = \begin{bmatrix}1 & 2 & 1 & 3 & 3\\2 & 3 & 1 & 7 & 4\\3 & 4 & 1 & 11 & 5\end{bmatrix}$ $r_2 \mapsto r_2 - 2r_1$, $r_3 \mapsto r_3 - 3r_1$

$\rightsquigarrow \begin{bmatrix}1 & 2 & 1 & 3 & 3\\0 & -1 & -1 & 1 & -2\\0 & -2 & -2 & 2 & -4\end{bmatrix}$ $r_3 \mapsto r_3 - 2r_2$ $\rightsquigarrow \begin{bmatrix}1 & 2 & 1 & 3 & 3\\0 & -1 & -1 & 1 & -2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$ $r_2 \mapsto -r_2$

$\rightsquigarrow \begin{bmatrix}1 & 2 & 1 & 3 & 3\\0 & 1 & 1 & -1 & 2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$ $r_1 \mapsto r_1 - 2r_2$ $\rightsquigarrow \begin{bmatrix}1 & 0 & -1 & 5 & -1\\0 & 1 & 1 & -1 & 2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$

$\Rightarrow V_1$ & V_2 span V and are linearly independent, hence $\mathcal{B} = (V_1, V_2)$ is a basis for V.

c. (4 points) Find the kernel of the matrix $A = [v_1\ v_2\ v_3\ v_4\ v_5]$.

By the computation in part b)

$rref(A) = \begin{bmatrix}1 & 0 & -1 & 5 & -1\\0 & 1 & 1 & -1 & 2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$ $\quad x_1 - x_3 + 5x_4 - x_5 = 0$, $\quad x_2 + x_3 - x_4 + 2x_5 = 0$

$\Rightarrow x_1 = r - 5s + t$ and $x_2 = -r + s - 2t$ with $x_3 = r$, $x_4 = s$, and $x_5 = t$, so

$$\ker A = \left\{ \begin{bmatrix}r - 5s + t\\-r + s - 2t\\r\\s\\t\end{bmatrix} \middle| \ r,s,t \in \mathbb{R} \right\} = \left\{ r\begin{bmatrix}1\\-1\\1\\0\\0\end{bmatrix} + s\begin{bmatrix}-5\\1\\0\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\-2\\0\\0\\1\end{bmatrix} \middle| \ r,s,t \in \mathbb{R} \right\}$$

$$= span\left(\begin{bmatrix}1\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-5\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\0\\0\\1\end{bmatrix} \right)$$

4. (10 points total) The following problem has parts a) and b):

a. (6 points) Find the 3×3 matrix which represents orthogonal projection onto the line spanned by $v = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$.

$\|v\| = \sqrt{1+9+16} = \sqrt{26}$

Let $V = \operatorname{span} v = \operatorname{span} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

$$T := \operatorname{Proj}_V x = \left(\frac{v}{\|v\|} \cdot x \right) \frac{v}{\|v\|} = \frac{1}{26} (v \cdot x)\, v$$

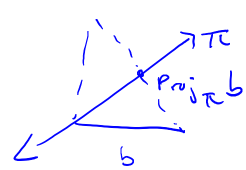
Recall that the matrix of a linear transformation has the form $\left[Te_1, Te_2, Te_3 \right]$, so

$$A = \left[\frac{1}{26}(v \cdot e_1)\, v \quad \frac{1}{26}(v \cdot e_2)\, v \quad \frac{1}{26}(v \cdot e_3)\, v \right] = \frac{1}{26} \begin{bmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ 4 & 12 & 16 \end{bmatrix}$$

b. (4 points) Find the matrix which represents reflection across the plane in $\mathbb{R}^3$ defined by the equation $x + 3y + 4z = 0$.

Projection onto the plane π is $(I - A) = B$ for A as in part a). Reflection is related to projection by

$$\operatorname{Ref}_\pi = I - 2(I - \operatorname{Proj}_\pi)$$
$$= 2 \operatorname{Proj}_\pi - I$$



$$= 2B - I = 2(I - A) - I = 2I - 2A - I$$
$$= I - 2A$$

$$\therefore \quad \operatorname{Ref}_\pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{13} \begin{bmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ 4 & 12 & 16 \end{bmatrix}$$

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