Math 33A: Midterm 2

2021 Spring

Please carefully read the following instructions:

- ☐ The exam will begin on May 10th at 8AM PT. You will be given **24 hours** to complete and submit your works. The submission window will be closed on May 11th at 8AM.
- ☐ **No late submission** will be considered. Make sure to spare enough time to complete and submit your solutions. Make-ups for the exam are permitted only under exceptional circumstances, as outlined in the UCLA student handbook.
- ☐ The exam will be **open book/open notes**. You can use any resources you find in our textbook or on our CCLE page.
- ☐ You must **show your works to receive credit.** Each of your solutions must clearly demonstrate all the key logical steps towards the answer. Partial credit will be scarce for incomplete solutions or answers without justification.
- ☐ You may use technology to write up your solutions, such as word processors or note-taking applications. You may also write your solutions on blank papers. If you choose to do so, please leave enough space between questions.
- ☐ A Gradescope link for submitting your work will be provided on the CCLE course webpage.
- ☐ If you have a question about the phrasing of the questions or about the exam logistics, you may email me (sos440@math.ucla.edu). Please make sure to begin the subject line of your email with the prefix 'Math 33A'; otherwise I will not reply to the email.
- ☐ You must **sign the code of conduct.** Any deviation from the rules will be considered as cheating. The university is also well-aware of "academic educational sites", and their use in connection with the exam is an Honor Code violation that is taken very seriously in UCLA.

Please read and sign the following honor code:

> *"I certify, on my honor, that I have not asked for or received assistance of any kind from any other person while working on the exam and that I have not used any non-permitted materials or technologies during the period of this evaluation."*

Name: [redacted]

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Q1. (8 points) Consider the vectors

$$\vec{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix}1\\k\\k^2\\k^3\end{bmatrix},$$

where k is a real number.

(a) For what value(s) of k do the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ fail to form a basis of $\mathbb{R}^4$?

By inspection, $\vec{v}_1\ \vec{v}_2\ \vec{v}_3$ are not redundant. So, check to see when $\vec{v}_4$ becomes redundant:

$$\vec{v}_4 = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = x_1\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + x_2\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\k\\k^2\\k^3\end{bmatrix} = \begin{bmatrix}x_1 + x_2\\ x_1 + x_3\\ x_1 + x_2 + x_3\\ x_1 + x_3\end{bmatrix}$$

Observe that $k = x_1 + x_3 = k^3$, so $k = k^3$. Therefore,

$(k^3 - k) = 0 \Rightarrow k = 0, 1, -1$

when $\boxed{k = 0, 1, -1}$ $\vec{v}_4$ is redundant and a basis is not formed.

(b) For each value of k found in part (a), find a non-trivial linear relation among $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

$k = 0$: $\quad c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$

$$\vec{v}_4 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{pmatrix}c_1 + c_2\\ c_1 + c_3\\ c_1 + c_2 + c_3\\ c_1 + c_3\end{pmatrix} \Rightarrow \left[\begin{array}{ccc|c}1&1&0&1\\1&0&1&0\\1&1&1&0\\1&0&1&0\end{array}\right] = A$$

rref(A):

$$\left[\begin{array}{ccc|c}1&1&0&1\\1&0&1&0\\1&1&1&0\\1&0&1&0\end{array}\right]\begin{array}{l}\\-R_1\\-R_1\\-R_1\end{array} \rightarrow \left[\begin{array}{ccc|c}1&1&0&1\\0&-1&1&-1\\0&0&1&-1\\0&-1&1&-1\end{array}\right]\begin{array}{l}+R_2\\ \times(-1)\\ \\ -R_2\end{array} \rightarrow \left[\begin{array}{ccc|c}1&0&1&0\\0&1&-1&1\\0&0&1&-1\\0&0&0&0\end{array}\right]\begin{array}{l}-R_3\\+R_3\\ \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&0\\0&0&1&-1\\0&0&0&0\end{array}\right]$$

so, $\vec{v}_4 = \vec{v}_1 + 0\,\vec{v}_2 - \vec{v}_3$

$= \vec{v}_1 - \vec{v}_3$

$$\boxed{-\vec{v}_1 + \vec{v}_3 + \vec{v}_4 = \vec{0} \qquad \begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix} = t\begin{bmatrix}-1\\0\\1\\1\end{bmatrix} \text{ for any } t \in \mathbb{R}}$$

(1B)

$k=1: \; c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$

$$A = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right]$$ is the augmented matrix

$$\text{rref}(A): \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array}\right]\begin{array}{l} \\ \times(-1) \\ \\ +R_2 \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} -R_2 \\ +R_3 \\ \\ \\ \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} -R_3 \\ \\ \\ \\ \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so, $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} t$ for any $t \in \mathbb{R}$

and $\vec{v}_4 = \vec{v}_1$, $-\vec{v}_1 + \vec{v}_4 = \vec{0}$

~~$k = -1$:~~

~~Augmented matrix for~~ $\cancel{A = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{array}\right]}$ ~~$-R_1$ $-R_1$ $-R_2$~~

~~rref A:~~ $\cancel{\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & \end{array}}$

$k = -1$:

$$A = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \end{array}\right]$$ is the aug matrix for $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$

$$\text{rref}(A): \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} +R_2 \\ \times(-1) \\ \\ \\ \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} -R_3 \\ +R_3 \\ \\ \\ \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} t$ for any $t \in \mathbb{R}$

$\vec{v}_4 = -\vec{v}_1 + 2\vec{v}_2 \Rightarrow \vec{v}_1 - 2\vec{v}_2 + \vec{v}_4 = \vec{0}$

$-5 + \frac{60}{5}$

$2 - \frac{18}{5}$ $= -5 + 12$

$= \frac{10-18}{5}$

$\frac{6}{5} - 2 = \frac{6-10}{5} = -4/5$

$\cancel{\frac{6}{5} - 8 = \frac{6-40}{5}}$

$2 - \frac{24}{5} = \frac{10-24}{5}$

Q2. (12 points) Let A be the 4×4 matrix given by

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 2 & 1 & 5 & -2 \\ -1 & 1 & -4 & 0 \\ 3 & -1 & 10 & 1 \end{bmatrix}.$$

(a) Find a basis of the image of A.

start by finding rref(A) :

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 2 & 1 & 5 & -2 \\ -1 & 1 & -4 & 0 \\ 3 & -1 & 10 & 1 \end{bmatrix} \begin{matrix} \\ -2R_1 \\ +R_1 \\ -3R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & -5 & 5 & -6 \\ 0 & 4 & -4 & 2 \\ 0 & -10 & 10 & -5 \end{bmatrix} \begin{matrix} \\ \div -5 \\ \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & 6/5 \\ 0 & 4 & -4 & 2 \\ 0 & -10 & 10 & -5 \end{bmatrix} \begin{matrix} -3R_2 \\ \\ -4R_1 \\ +10R_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & -8/5 \\ 0 & 1 & -1 & 6/5 \\ 0 & 0 & 0 & -14/5 \\ 0 & 0 & 0 & 7 \end{bmatrix} \begin{matrix} \\ \\ (\times -5/4) \\ \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -8/5 \\ 0 & 1 & -1 & 6/5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix} \begin{matrix} \\ \\ \\ -7R_3 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & -8/5 \\ 0 & 1 & -1 & 6/5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} +8/5 R_3 \\ -6/5 R_3 \\ \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

column 1, 2, 4 contain leading ones, so they will form a basis of im(A)

$$\boxed{\therefore \text{ basis of } im(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

(b) Find a basis of the kernel of A.

By inspection: column 3 is a redundant vector because it did not contain a leading one in rref(A).

$$\vec{v}_3 = 3\vec{v}_1 - \vec{v}_2$$

$$-3\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$$

By Theorem 3.3.8,

$$\boxed{\text{basis of } \ker(A) = \left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

(c) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ be the column vectors of A in the order as they appear. Now we consider a vector $\vec{v}_5$ in $\mathbb{R}^4$ and form a 4×5 matrix B by appending $\vec{v}_5$ to the right of A, i.e.,

$$B = \begin{bmatrix} | & | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \\ | & | & | & | & | \end{bmatrix}.$$

Find all possible values of $\vec{v}_5$ such that the image of B is a 3-dimensional subspace of $\mathbb{R}^4$. (*Hint:* Essentially no computation is needed in this part.)

since im(A) is already a 3dimensional subspace of $\mathbb{R}^4$, $\vec{v}_5$ must be a redundant vector and a linear combination of $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_4$.

$$\vec{v}_5 = r\vec{v}_1 + s\vec{v}_2 + t\vec{v}_4$$

$$\boxed{\vec{v}_5 = r\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + s\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \text{ for any } r, s, t \in \mathbb{R}}$$

Q3. (12 points) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be the vectors in $\mathbb{R}^3$ given by

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

You are given that these three vectors form a basis of $\mathbb{R}^3$.

(a) Perform the Gram–Schmidt process on the list of vectors $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$. Make sure to provide all the details of your computation.

1. compute $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{\sqrt{4+9+36}} \vec{v}_1 = \frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$

2. compute $\vec{u}_2 = \frac{1}{\|\vec{v}_2^\perp\|} \vec{v}_2^\perp$ using $V_1 = \text{span}(\vec{u}_1)$

$\rightarrow \vec{v}_2^\perp = \vec{v}_2 - \text{proj}_{V_1}(\vec{v}_2) = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$

$= \vec{v}_2 - \left(\frac{2}{7} + 0 + \frac{12}{7}\right)\vec{u}_1 = \vec{v}_2 - 2\vec{u}_1$

$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{2}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 - 4/7 \\ 0 - 6/7 \\ 2 - 12/7 \end{bmatrix} = \begin{bmatrix} 3/7 \\ -6/7 \\ 2/7 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$

since we only care about the direction of $\vec{v}_2^\perp$, we can drop $\frac{1}{7}$ and use $\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$:

$\|\vec{v}_2^\perp\| = \sqrt{9+36+4} = 7$

$\vec{u}_2 = \frac{1}{7}\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$ ($\vec{v}_2^\perp$ was already a unit vector)

3. compute $\vec{u}_3 = \frac{1}{\|\vec{v}_3^\perp\|} \vec{v}_3^\perp$, $\text{span}(\vec{u}_1, \vec{u}_2) = V_2$

$\rightarrow \vec{v}_3^\perp = \vec{v}_3 - \text{proj}_{V_2}(\vec{v}_3)$

$= \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2$ $\qquad 1 - \frac{9}{7} = \frac{-2}{7}$

$= \vec{v}_3 - \left(-\frac{2}{7} + \frac{3}{7} + \frac{6}{7}\right)\vec{u}_1 - \left(-\frac{3}{7} - \frac{6}{7} + \frac{2}{7}\right)\vec{u}_2$ $\qquad 1 - \frac{4}{7} = 3/7$

$= \vec{v}_3 - (\vec{u}_1) - (-1)\vec{u}_2$

$= \vec{v}_3 - \vec{u}_1 + \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/7 \\ 3/7 \\ 6/7 \end{bmatrix} + \begin{bmatrix} 3/7 \\ -6/7 \\ 2/7 \end{bmatrix}$

$= \begin{bmatrix} -6/7 \\ -2/7 \\ 3/7 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -6 \\ -2 \\ 3 \end{bmatrix}$ is already a unit vector.

Note that $\vec{u}_3 = \vec{v}_3^\perp$ because $\vec{v}_3^\perp$ is already a unit vector.

$$\boxed{\therefore \left\{ \frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \frac{1}{7}\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}, \frac{1}{7}\begin{bmatrix} -6 \\ -2 \\ 3 \end{bmatrix} \right\} \text{ form an ONB of } \mathbb{R}^3}$$

(b) Let $\mathfrak{B} = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$ be the orthonormal basis that is obtained as the outcome of the Gram–Schmidt process in part (a). Find $[\vec{v}_3]_{\mathfrak{B}}$, the coordinate vector of $\vec{v}_3$ with respect to $\mathfrak{B}$.

$$\vec{v}_3^{\perp} = \vec{v}_3 - \text{proj}_{V_2}(\vec{v}_3)$$

(note that $\vec{v}_3^{\perp} = \vec{u}_3$, from part A)

$$\vec{v}_3 = \vec{v}_3^{\perp} + \text{proj}_{V_2}(\vec{v}_3)$$

$$= \vec{u}_3 + \left[(\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 + (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2\right]$$

(by theorem 5.1.5)

$$= \vec{u}_3 + \left[\left(-\tfrac{2}{7} + \tfrac{3}{7} + \tfrac{6}{7}\right)\vec{u}_1 + \left(-\tfrac{3}{7} - \tfrac{6}{7} + \tfrac{2}{7}\right)\vec{u}_2\right]$$

$$= \vec{u}_3 + \vec{u}_1 + (-1)\vec{u}_2$$

$$= \vec{u}_1 + (-1)\vec{u}_2 + \vec{u}_3$$

$$\therefore \boxed{[\vec{v}_3]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$$

(3c)

By inspection:

$T(\vec{u}_1) = \vec{u}_1$ (since $\vec{u}_1$ and $\vec{u}_2$ lie in V)

$T(\vec{u}_2) = \vec{u}_2$

$$\begin{aligned} T(\vec{u}_3) &= 2\,\mathrm{proj}_V(\vec{u}_3) - \vec{u}_3 \\ &= 2\left[(\vec{u}_1 \cdot \vec{u}_3)\vec{u}_1 + (\vec{u}_2 \cdot \vec{u}_3)\vec{u}_2\right] - \vec{u}_3 \\ &= 2\left[0 + 0\right] - \vec{u}_3 \\ &= -\vec{u}_3 \end{aligned}$$

(since $\vec{u}_3$ is perpendicular to V)

so, $[T(\vec{u}_1)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$[T(\vec{u}_2)]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$[T(\vec{u}_3)]_B = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

B-matrix of T is $\boxed{B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$ by theorem 3.4.3.

Q4. (8 points) Consider the 3×3 matrix A and the vector $\vec{b} \in \mathbb{R}^3$ given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Find all the least-squares solutions $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$.

using theorem D15.8: $A^TA\vec{x}^* = A^T\vec{b}$

since $\ker(A) \neq \{\vec{0}\}$, there is not a unique solution.

→ compute $A^TA = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 9 & 7 \\ 9 & 9 & 9 \\ 7 & 9 & 11 \end{bmatrix}$

$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix}$

so, we have:

$$\begin{bmatrix} 11 & 9 & 7 \\ 9 & 9 & 9 \\ 7 & 9 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix}$$

Augmented matrix: $\left[\begin{array}{ccc|c} 11 & 9 & 7 & 9 \\ 9 & 9 & 9 & 9 \\ 7 & 9 & 11 & 9 \end{array}\right]$

rref: $\left[\begin{array}{ccc|c} 1 & 9/11 & 7/11 & 9/11 \\ 1 & 1 & 1 & 1 \\ 7 & 9 & 11 & 9 \end{array}\right] \begin{matrix} \\ -R_1 \\ -7R_1 \end{matrix} \rightarrow \left[\begin{array}{ccc|c} 1 & 9/11 & 7/11 & 9/11 \\ 0 & 2/11 & 4/11 & 2/11 \\ 0 & 36/11 & 72/11 & 36/11 \end{array}\right] \begin{matrix} \times 11 \\ \times 11 \\ \times 11 \end{matrix} \rightarrow \left[\begin{array}{ccc|c} 11 & 9 & 7 & 9 \\ 0 & 2 & 4 & 2 \\ 0 & 36 & 72 & 36 \end{array}\right] \div 2$

$\left[\begin{array}{ccc|c} 11 & 9 & 7 & 9 \\ 0 & 1 & 2 & 1 \\ 0 & 36 & 72 & 36 \end{array}\right] \begin{matrix} -9R_2 \\ \\ -36R_2 \end{matrix} \rightarrow \left[\begin{array}{ccc|c} 11 & 0 & -11 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \div 11 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

so, $x - z = 0$, $y + 2z = 1$, z is free $\Rightarrow$ $x = z$, $y = -2z + 1$, z free $\Rightarrow$ $\boxed{\vec{x}^* = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ for any } t \in \mathbb{R}}$

(b) Use the answer in part (a) to find the minimum distance between $\vec{b}$ and any vector $\vec{v}$ in $\mathrm{im}(A)$.

$\|-\vec{b} + A\vec{x}^*\| \leq \|-\vec{b} + A\vec{v}\|$ for all $\vec{v} \in \mathrm{im}(A)$, by definition

$A\vec{x}^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} t \\ -2t+1 \\ t \end{bmatrix} = \begin{bmatrix} t + (-2t+1) + t \\ t + 2(-2t+1) + 3t \\ 3t + 2(-2t+1) + t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$A\vec{x}^* - \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ 2 \end{bmatrix}$

$\|A\vec{x}^* - \vec{b}\| = \sqrt{64 + 4 + 4} = \sqrt{72} = \boxed{6\sqrt{2}}$

Q5. (9 points) For each of the following problems, provide an example with the given properties, and briefly explain why your choice of example works. If no such example exists, then prove why it is impossible to find such one.

(a) An example of a 3×3 matrix A such that both $\text{im}(A)$ and $\ker(A)$ are planes in $\mathbb{R}^3$.

False. matrix A with the given properties does not exist by the Rank-Nullity Theorem. A plane is a 2 dimensional subspace. If im(A) and ker(A) are planes in $\mathbb{R}^3$, then $\dim(\text{im}(A)) = 2$ and $\dim(\ker(A)) = 2$.

However, by the Rank-Nullity theorem,

$$\dim(\text{im}(A)) + \dim(\ker(A)) = m$$

since $2 + 2 \neq 3$

$4 \neq 3$,

$\boxed{\text{matrix A does not exist.}}$

(b) An example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\|T(\vec{x})\| = \|\vec{x}\| \quad \text{and} \quad \vec{x} \cdot T(\vec{x}) = 0$$

perpendicular $\vec{x} \perp T(\vec{x})$

for any $\vec{x}$ in $\mathbb{R}^2$.

Yes. The rotation of 90° either clockwise or counterclockwise satisfies the given conditions. Rotations preserve length, and rotating a vector through an angle of 90° will produce a vector that is perpendicular to the original vector.

example: $T(\vec{x}) = \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} \vec{x}$

$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$

proof: $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$

$\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \sqrt{x^2+y^2}$, $\left\| \begin{bmatrix} -y \\ x \end{bmatrix} \right\| = \sqrt{y^2+x^2} = \sqrt{x^2+y^2}$ ✓

$\begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} -y \\ x \end{bmatrix} = -xy + xy = 0$ ✓

$\boxed{\therefore T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x} \text{ satisfies the conditions.}}$

(c) An example of a 3×3 matrix P which is not a scalar multiple of I_3 and satisfies $P^2 = P$.

$P^2 = P$

Geometrically, we can interpret P as the matrix of an orthogonal projection onto a subspace V of $\mathbb{R}^3$. Since $P\vec{x}$ is the projection of $\vec{x}$ onto V, $P\vec{x}$ lies in V. Applying the projection once more will yield $P\vec{x}$ itself, since $P\vec{x}$ already lies in V. Thus, $P(P\vec{x}) = P^2\vec{x} = P\vec{x}$.

example :

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$V = \text{span}(\vec{e}_1, \vec{e}_2)$ form a subspace of $\mathbb{R}^3$ and $\vec{e}_1$ and $\vec{e}_2$ form an orthonormal basis of V.

Apply Theorem D14.9 (Matrix of orthogonal projection):

$$\text{proj}_V(\vec{x}) = QQ^T\vec{x}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad Q^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P = QQ^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

check that $P^2 = P$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \checkmark$$

so, $\boxed{P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$

Q6. (9 points) Prove each of the following statements.

(a) If V is a subspace of $\mathbb{R}^n$ and $\vec{x}$ is any vector in $\mathbb{R}^n$, then $\|\operatorname{proj}_V(\vec{x})\| \leq \|\vec{x}\|$.

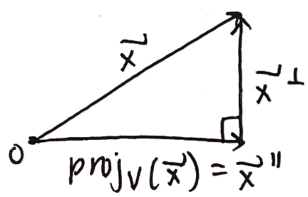
Apply the pythagorean Theorem:

$$\|\vec{x}\|^2 = \|\operatorname{proj}_V \vec{x}\|^2 + \|\vec{x}^{\perp}\|^2$$

$$\|\operatorname{proj}_V \vec{x}\|^2 = \|\vec{x}\|^2 - \|\vec{x}^{\perp}\|^2$$

(using $\vec{x} = \vec{x}^{\perp} + \vec{x}^{\parallel}$)

$\therefore$ $\|\operatorname{proj}_V \vec{x}\| = \|\vec{x}\|$ only when $\vec{x}$ lies in V.
otherwise, $\|\operatorname{proj}_V \vec{x}\| < \|\vec{x}\|$ since $\|\vec{x}^{\perp}\|$ is always positive.



Hypotenuse of a right triangle is always the longest side.

(b) If Q is a 3×3 orthogonal matrix and $\vec{u}_1, \vec{u}_2, \vec{u}_3$ form an orthonormal basis of $\mathbb{R}^3$, then $Q\vec{u}_1, Q\vec{u}_2, Q\vec{u}_3$ also form an orthonormal basis of $\mathbb{R}^3$.

Let $A = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}$. Then, A is a 3×3 orthogonal matrix because its columns form an ONB of $\mathbb{R}^3$ (theorem 5.3.3).

We know that the columns of Q form an ONB of $\mathbb{R}^3$, also by theorem 5.3.3.

So, consider their product: $QA = Q\begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}$

By theorem 2.3.2, $QA = \begin{bmatrix} | & | & | \\ Q\vec{u}_1 & Q\vec{u}_2 & Q\vec{u}_3 \\ | & | & | \end{bmatrix}$

By theorem 5.3.4, QA is orthogonal because both Q and A are orthogonal.

So, QA is an orthogonal matrix. Since the column vectors of an orthogonal matrix form an ONB (5.3.3),

$\boxed{Q\vec{u}_1, Q\vec{u}_2, Q\vec{u}_3 \text{ must form an orthonormal basis of } \mathbb{R}^3.}$

$\nearrow$ 2021×1

A is a 3×2021 matrix

(c) If $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are vectors in $\mathbb{R}^{2021}$, and if there exists a linear transformation $T : \mathbb{R}^{2021} \to \mathbb{R}^3$ such that

$$T(\vec{v}_1) = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad T(\vec{v}_2) = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \text{and} \quad T(\vec{v}_3) = \begin{bmatrix}0\\0\\1\end{bmatrix},$$

then the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are linearly independent.

suppose $\vec{V}_1$, $\vec{V}_2$, $\vec{V}_3$ are <u>not</u> linearly independent.

Then, there exists a non-trivial relationship among $\vec{V}_1$, $\vec{V}_2$, $\vec{V}_3$:

$C_1 \vec{V}_1 + C_2 \vec{V}_2 + C_3 \vec{V}_3 = \vec{0}$ where some of $C_1, C_2, C_3 \neq 0$.

apply the transformation T to both sides:

$T(C_1 \vec{V}_1 + C_2 \vec{V}_2 + C_3 \vec{V}_3) = T(\vec{0})$

$T(C_1 \vec{V}_1 + C_2 \vec{V}_2 + C_3 \vec{V}_3) = \vec{0}$

$T(C_1 \vec{V}_1) + T(C_2 \vec{V}_2) + T(C_3 \vec{V}_3) = \vec{0}$ using linearity of T

$C_1 T(\vec{V}_1) + C_2 T(\vec{V}_2) + C_3 T(\vec{V}_3) = \vec{0}$

Then, if some of $C_1, C_2, C_3 \neq 0$, $T(\vec{V}_1)$, $T(\vec{V}_2)$, $T(\vec{V}_3)$ would <u>NOT</u> be linearly independent, ie some of $T(\vec{V}_1)$, $T(\vec{V}_2)$, $T(\vec{V}_3)$ would be <u>redundant</u>.

However, it can be seen by inspection that $T(\vec{V}_1) = \vec{e}_1$, $T(\vec{V}_2) = \vec{e}_2$, $T(\vec{V}_3) = \vec{e}_3$ <u>are</u> linearly independent.

Thus, the assumption that $\vec{V}_1$, $\vec{V}_2$, $\vec{V}_3$ are linearly dependent was <u>false</u>. so, $\vec{V}_1$, $\vec{V}_2$, and $\vec{V}_3$

<u>are linearly independent.</u>