Math 33A: Final
2021 Spring

Please carefully read the following instructions:

- ☐ The exam will begin on June 6th at 8AM PT. You will be given **24 hours** to complete and submit your works. The submission window will be closed on June 7th at 8AM.
- ☐ **No late submission** will be considered. Make sure to spare enough time to complete and submit your solutions. Make-ups for the exam are permitted only under exceptional circumstances, as outlined in the UCLA student handbook.
- ☐ The exam will be **open book/open notes**. You can use any resources you find in our textbook or on our CCLE page. You may use technology to compute matrix multiplication and matrix inversion, unless specified otherwise.
- ☐ You must **show your works to receive credit.** Each of your solutions must clearly demonstrate all the key logical steps towards the answer. Partial credit will be scarce for incomplete solutions or answers without justification.
- ☐ You may use technology to write up your solutions, such as word processors or note-taking applications. You may also write your solutions on blank papers. If you choose to do so, please leave enough space between questions.
- ☐ A Gradescope link for submitting your work will be provided on the CCLE course webpage.
- ☐ If you have a question about the phrasing of the questions or about the exam logistics, you may email me (sos440@math.ucla.edu). Please make sure to begin the subject line of your email with the prefix 'Math 33A'; otherwise I will not reply to the email.
- ☐ You must **sign the code of conduct.** Any deviation from the rules will be considered as cheating. The university is well-aware of "academic educational sites", and their use in connection with the exam is an Honor Code violation that is taken very seriously in UCLA.

Please read and sign the following honor code:

> *"I certify, on my honor, that I have not asked for or received assistance of any kind from any other person while working on the exam and that I have not used any non-permitted materials or technologies during the period of this evaluation."*

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UID: [redacted]

Signatur[redacted]

Q1. (9 points) This question consists of several, unrelated sub-questions regarding determinant. Answer each of the sub-questions and provide a brief explanation.

(a) What is the determinant of the matrix A given by

$$A = \begin{bmatrix} 7 & 0 & 0 \\ 5 & 6 & 0 \\ 3 & -8 & 2 \end{bmatrix} \begin{bmatrix} 3 & 9 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & -1 \end{bmatrix}^{-1} \quad ?$$

Let $B = \begin{bmatrix} 3 & 9 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & -1 \end{bmatrix}$. We check that B is invertible by computing its determinant using THM 6.1.4, since B is an upper triangular matrix. $\det(B)$ = product of diagonal entries $= (3)(4)(-1) = -12$.

$-12 \neq 0$, so B is invertible and we find $\det(B^{-1})$ using THM 6.2.8.

$\det(B^{-1}) = \frac{1}{\det B} = \frac{1}{-12} = -\frac{1}{12}$.

Using THM 6.2.6, we compute $\det(A)$; let $C = \begin{bmatrix} 7 & 0 & 0 \\ 5 & 6 & 0 \\ 3 & -8 & 2 \end{bmatrix}$

$$\det(A) = \det(CB^{-1}) = \det(C) \cdot \det(B^{-1})$$
$$= (7 \cdot 6 \cdot 2)\left(-\frac{1}{12}\right) = \boxed{-7}$$

where we have applied THM 6.1.4 to find $\det(C)$ since C is lower triangular.

(b) Let A be the matrix of the orthogonal projection onto the plane $2x - 3y + z = 0$ in $\mathbb{R}^3$. Find the determinant of A.

Note that this linear transformation is NOT injective — many different vectors can map to the same projection in the plane (ie, all vectors $\perp$ to the plane will be mapped to the zero vector). Thus, the linear transformation is NOT invertible and has $\boxed{\text{determinant} = 0}$

(c) Suppose we are given that

A

$$\det \begin{bmatrix} 1 & 1 & 1 & a \\ 1 & 2 & 0 & b \\ 1 & 3 & 0 & c \\ 1 & 4 & 0 & d \end{bmatrix} = 15$$

holds for constants a, b, c, and d. What is the value of

B

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3a & 3b & 3c & 3d \\ 1 & 0 & 0 & 0 \\ 3 & 5 & 7 & 9 \end{bmatrix} \quad ?$$

If $A = \begin{bmatrix} 1 & 1 & 1 & a \\ 1 & 2 & 0 & b \\ 1 & 3 & 0 & c \\ 1 & 4 & 0 & d \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 \\ a & b & c & d \end{bmatrix}$ and $\det(A^T) = \det(A) = 15$ by THM 6.2.1.

transform A^T into B:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 \\ a & b & c & d \end{bmatrix} \xrightarrow[d=-1]{} \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} \times 3 \xrightarrow[d=\frac{1}{3}]{} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3a & 3b & 3c & 3d \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} \times 2$$

$$\xrightarrow[d=\frac{1}{2}]{} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3a & 3b & 3c & 3d \\ 1 & 0 & 0 & 0 \\ 2 & 4 & 6 & 8 \end{bmatrix} + R_1 \xrightarrow[d=1]{} B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3a & 3b & 3c & 3d \\ 1 & 0 & 0 & 0 \\ 3 & 5 & 7 & 9 \end{bmatrix}$$

Then,

$$\det(A^T) = (-1)\left(\tfrac{1}{3}\right)\left(\tfrac{1}{2}\right)(1)\det(B)$$

$$15 = -\tfrac{1}{6}\det(B)$$

$$\boxed{\det(B) = -90} \quad \text{where } B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3a & 3b & 3c & 3d \\ 1 & 0 & 0 & 0 \\ 3 & 5 & 7 & 9 \end{bmatrix}$$

Q2. (9 points) Consider the matrix

$$A = \begin{bmatrix} 4 & -4 & 2 \\ 0 & 1 & 0 \\ -3 & 4 & -1 \end{bmatrix}.$$

(a) Compute the characteristic polynomial

$$f_A(\lambda) = \det(A - \lambda I_3),$$

and then find the eigenvalues for A. In doing so, do not use technology and briefly explain what method or result you are using to compute $f_A(\lambda)$.

$$A - \lambda I_3 = \begin{bmatrix} 4-\lambda & -4 & 2 \\ 0 & 1-\lambda & 0 \\ -3 & 4 & -1-\lambda \end{bmatrix},$$ compute $\det(A - \lambda I_3)$ using Sarrus's Rule.

$$\det(A - \lambda I_3) = \left[\begin{array}{ccc|cc} 4-\lambda & -4 & 2 & 4-\lambda & -4 \\ 0 & 1-\lambda & 0 & 0 & 1-\lambda \\ -3 & 4 & -1-\lambda & -3 & 4 \end{array}\right]$$

$$= [(4-\lambda)(1-\lambda)(-1-\lambda) + 0 + 0] - [-6(1-\lambda) + 0 + 0]$$

$$= (4 - 5\lambda + \lambda^2)(-1-\lambda) + 6(1-\lambda) = -\lambda^3 + 4\lambda^2 + \lambda - 4 + 6 - 6\lambda$$

$$= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = (1-\lambda)(\lambda-1)(\lambda-2)$$

so, $\boxed{f_A(\lambda) = \det(A - \lambda I_3) = (1-\lambda)(\lambda-1)(\lambda-2) = -(\lambda-1)^2(\lambda-2)}$

and the eigenvalues are obtained by setting $f_A(\lambda) = 0$:

$$-(\lambda-1)^2(\lambda-2) = 0$$

$$\boxed{\begin{array}{cc} \lambda_1 = 1, & \lambda_2 = 2 \\ \text{almu}(1) = 2, & \text{almu}(2) = 1 \end{array}}$$

(b) For each eigenvalue λ of A, find the eigenspace E_λ.

- $E_1 = \ker(A - I_3) = \ker \begin{bmatrix} 3 & -4 & 2 \\ 0 & 0 & 0 \\ -3 & 4 & -2 \end{bmatrix} = \ker \begin{bmatrix} 3 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

so, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in E_1 \longleftrightarrow 3x_1 - 4x_2 + 2x_3 = 0$ (let $x_2 = 3s$ and $x_3 = 3t$)

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$ $s, t \in \mathbb{R}$

so, $\boxed{\left(\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \right) \text{ is a basis of } E_1}$, with dim = 2.

- $E_2 = \ker(A - 2I_3) = \ker \begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 0 \\ -3 & 4 & -3 \end{bmatrix} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$ by inspection

$\therefore \boxed{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ is a basis of } E_2, \text{ with dim} = 1}$

(c) Determine whether A is diagonalizable or not, and briefly explain why. Also, if A is diagonalizable, then find an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$ holds.

since gemu(1) + gemu(2) = dim(E_1) + dim(E_2)

= 2 + 1 = 3 and A is 3x3 matrix,

A is diagonalizable. Another way to check this is:

gemu(1) = 2 = almu(1) and gemu(2) = 1 = almu(2).

$$\boxed{S = \begin{bmatrix} 4 & -2 & 1 \\ 3 & 0 & 0 \\ 0 & 3 & -1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ s/t } S^{-1}AS = D \text{ holds}}$$

Q3. (9 points) Consider the transition matrix A and the distribution vector $\vec{x}_0$ given by

$$A = \begin{bmatrix} 0.5 & 0.2 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0.1 & 0.6 & 0.2 \end{bmatrix} \quad \text{and} \quad \vec{x}_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

(a) Find an eigenvector $\vec{v}_1$ of A with associated eigenvalue 1.

$$E_1 = \ker(A - I_3) = \ker \begin{bmatrix} -0.5 & 0.2 & 0.4 \\ 0.4 & -0.8 & 0.4 \\ 0.1 & 0.6 & -0.8 \end{bmatrix}$$

put into rref:
$$\begin{bmatrix} -5 & 2 & 4 \\ 4 & -8 & 4 \\ 1 & 6 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & -8 \\ 4 & -8 & 4 \\ -5 & 2 & 4 \end{bmatrix} \begin{matrix} \\ -4R_1 \\ +5R_1 \end{matrix} \to \begin{bmatrix} 1 & 6 & -8 \\ 0 & -32 & 36 \\ 0 & 32 & -36 \end{bmatrix} \begin{matrix} \\ \\ +R_2 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 6 & -8 \\ 0 & -32 & 36 \\ 0 & 0 & 0 \end{bmatrix} \div -32 \to \begin{bmatrix} 1 & 6 & -8 \\ 0 & 1 & -\frac{9}{8} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-6R_2} \begin{bmatrix} 1 & 0 & -5/4 \\ 0 & 1 & -9/8 \\ 0 & 0 & 0 \end{bmatrix}$$

so, $\ker(A - I_3) = \operatorname{span}\begin{bmatrix} 5/4 \\ 9/8 \\ 1 \end{bmatrix}$

Then, an eigenvector w/ associated $\lambda = 1$ is $\boxed{\vec{v}_1 = \begin{bmatrix} 10 \\ 9 \\ 8 \end{bmatrix}}$

(b) Find a closed-form expression for

$$A^t \vec{x}_0,$$

where t is any positive integer. In doing so, feel free to use the fact that $A\vec{v}_2 = (-0.2)\vec{v}_2$ and $A\vec{v}_3 = (0.1)\vec{v}_3$ hold for the vectors $\vec{v}_2$ and $\vec{v}_3$ given by

$\lambda_2 = -0.2$
$\lambda_3 = 0.1$

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix} \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$\vec{x}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$, $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = [\vec{x}_0]_{\mathcal{B}} = S^{-1}\vec{x}_0 = \begin{bmatrix} 10 & 2 & 1 \\ 9 & 3 & 0 \\ 8 & -5 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/27 \\ 0 \\ -1/27 \end{bmatrix}$$

(computed using technology)

Then, $A^t \vec{x}_0 = c_1 A^t \vec{v}_1 + c_2 A^t \vec{v}_2 + c_3 A^t \vec{v}_3$

$$= \frac{1}{27}\begin{bmatrix} 10 \\ 9 \\ 8 \end{bmatrix} + 0 \cdot (-0.2)^t \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix} - \frac{1}{27} \cdot (0.1)^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \boxed{\frac{1}{27}\begin{bmatrix} 10 \\ 9 \\ 8 \end{bmatrix} - \frac{1}{27}(0.1)^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$$

(c) Find $\lim_{t\to\infty} A^t\vec{x}_0$.

$\lim_{t\to\infty} A^t \vec{x}_0$ using $A^t\vec{x}_0$ from part B:

$$= \boxed{\frac{1}{27}\begin{bmatrix}10\\9\\8\end{bmatrix} = \vec{x}_{equ}}$$

since $(0.1)^t$ goes to 0 as $t\to\infty$, so the second term in $A^t\vec{x}_0$ disappears

Q4. (9 points) Consider the matrix

$$A = \begin{bmatrix} -3 & 4 & -3 \\ 0 & 2 & 0 \\ 3 & 4 & 3 \end{bmatrix}.$$

(a) Find the singular values $\sigma_1, \sigma_2, \sigma_3$ of A.

1. $A^TA = \begin{bmatrix} -3 & 0 & 3 \\ 4 & 2 & 4 \\ -3 & 0 & 3 \end{bmatrix}\begin{bmatrix} -3 & 4 & -3 \\ 0 & 2 & 0 \\ 3 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 0 & 18 \\ 0 & 36 & 0 \\ 18 & 0 & 18 \end{bmatrix}$ (computed w/ technology)

2. $f_{ATA}(\lambda) = \det \begin{bmatrix} 18-\lambda & 0 & 18 \\ 0 & 36-\lambda & 0 \\ 18 & 0 & 18-\lambda \end{bmatrix} \Rightarrow$ (sarrus's rule) $\left| \begin{matrix} 18-\lambda & 0 & 18 \\ 0 & 36-\lambda & 0 \\ 18 & 0 & 18-\lambda \end{matrix} \right| \begin{matrix} 18-\lambda & 0 \\ 0 & 36-\lambda \\ 18 & 0 \end{matrix}$

singular values are square root of eigenvalues of A^TA

$= (18-\lambda)^2(36-\lambda) - 18^2(36-\lambda)$

$= (\cancel{18^2} - 36\lambda + \lambda^2 - \cancel{18^2})(36-\lambda) = (36-\lambda)(\lambda^2 - 36\lambda)$

$= \lambda(\lambda-36)(36-\lambda) = -\lambda(\lambda-36)^2$

$\Rightarrow \lambda_1 = 36, \lambda_2 = 0 \Rightarrow \boxed{\sigma_1 = 6, \sigma_2 = 6, \sigma_3 = 0}$

(b) For each eigenvalue λ of A^TA, find an orthonormal basis of the associated eigenspace E_λ.

- $E_{36} = \ker(ATA - 36I_3) = \ker \begin{bmatrix} -18 & 0 & 18 \\ 0 & 0 & 0 \\ 18 & 0 & -18 \end{bmatrix} = \ker \begin{bmatrix} -18 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

by inspection, $= \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$ $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

using the Gram-schmidt process,

$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 0\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

so, $\boxed{\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) \text{ is an ONB of } E_{36}}$

- $E_0 = \ker(A^TA) = \ker \begin{bmatrix} 18 & 0 & 18 \\ 0 & 36 & 0 \\ 18 & 0 & 18 \end{bmatrix} = \text{span}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ by inspection

make $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ a unit vector:

$\boxed{\vec{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ is an ONB of } E_0}$

(c) Find a singular value decomposition (SVD) for A.

$$\Sigma = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ (the non-zero singular values of A are the diagonal entries)

$$V = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

compute U:

$$U = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}$$ where $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ form an ONB of $\mathbb{R}^3$.

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{6}\begin{bmatrix} -3 & 4 & -3 \\ 0 & 2 & 0 \\ 3 & 4 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \frac{1}{6}\begin{bmatrix} -3 & 4 & -3 \\ 0 & 2 & 0 \\ 3 & 4 & 3 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -3/\sqrt{2} \\ 0 \\ 3/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

~~Note that $\vec{w}_2$ is not a unit vector, so we multiply it by a factor of 2 later, to get $\vec{u}_2$~~

we can't use the formula $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$ to compute $\vec{u}_3$ because $\sigma_3 = 0$.

so, pick a vector $\vec{u}_3$ such that $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ form an ONB of $\mathbb{R}^3$:

~~$\vec{u}_2 = 2\vec{w}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$~~

compute $\vec{u}_1 \times \vec{u}_2$:
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2/3 & 1/3 & 2/3 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 2/3 & 1/3 \\ \frac{-1}{\sqrt{2}} & 0 \end{matrix}$$

$$= \left(\frac{1}{3\sqrt{2}}\vec{i} - \frac{2}{3\sqrt{2}}\vec{j} + 0\vec{k}\right) - \left(0\vec{i} + \frac{2}{3\sqrt{2}}\vec{j} - \frac{1}{3\sqrt{2}}\vec{k}\right)$$

$$= \frac{1}{3\sqrt{2}}\vec{i} - \frac{4}{3\sqrt{2}}\vec{j} + \frac{1}{3\sqrt{2}}\vec{k}$$

$$\vec{v}_3 = \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} = \vec{u}_3 \cancel{= \frac{1}{\|\vec{v}_3\|}\vec{v}_3 = \frac{2}{1}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix} 1/6 \\ -2/3 \\ 1/6 \end{bmatrix} = \frac{2}{\sqrt{2}}\begin{bmatrix} 1/6 \\ -2/3 \\ 1/6 \end{bmatrix} = \sqrt{2}\begin{bmatrix} 1/6 \\ -2/3 \\ 1/6 \end{bmatrix}}$$

since $\vec{v}_3$ is a unit vector

$$\text{so, } U = \begin{bmatrix} 2/3 & -\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ 1/3 & 0 & -4/3\sqrt{2} \\ 2/3 & 1/\sqrt{2} & \frac{1}{3\sqrt{2}} \end{bmatrix}$$

and
$$\boxed{A = U\Sigma V^T = \begin{bmatrix} 2/3 & -1/\sqrt{2} & \sqrt{2}/6 \\ 1/3 & 0 & -2\sqrt{2}/3 \\ 2/3 & 1/\sqrt{2} & \sqrt{2}/6 \end{bmatrix}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T}$$

Q5. (9 points) For each of the following problems, provide an example with the given properties, and briefly explain why your choice of example works. If no such example exists, then explain why it is impossible to find such one.

(a) A 2×2 matrix A such that $A \neq I_2$ and the singular values of A are $\sigma_1 = \sigma_2 = 1$.

Any 2x2 orthogonal matrix will work, since $A^TA = I_n$ for orthogonal matrices by THM D14.8, and the eigenvalues of I_2 are 1 by THM D19.7. Since $\sigma = \sqrt{\lambda}$, $\sigma = \sqrt{1} = 1$.

example: $\boxed{A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$

$$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1$ (it is a diagonal matrix, so its eigenvalues are its diagonal entries)

$$\therefore \sigma_1 = \sqrt{\lambda_1} = \sqrt{1} = 1$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$$

$$\boxed{\sigma_1 = \sigma_2 = 1}$$

(b) A symmetric 3×3 matrix A such that A^2 is negative definite.

all eigen values are negative

$$q(\vec{x}) = \vec{x}^T(A^2)\vec{x}$$

$$= \vec{x}^T(AA)\vec{x}$$

$$= \vec{x}^T(AA^T\vec{x})$$

since $A = A^T$ for symmetric matrices

$$= (\vec{x}^TA)(A^T\vec{x})$$

No such example exists. Since A is symmetric, then A is orthogonally diagonalizable by the spectral Theorem. so, $A = SDS^{-1}$ for a diagonal D and orthogonal S. Then, $A^2 = (SDS^{-1})(SDS^{-1}) = SD\cancel{S^{-1}S}DS^{-1} = SD^2S^{-1}$ and A^2 cannot be negative definite because its eigenvalues are the squares of the eigenvalues of A. This proves that A^2 only has non-negative eigenvalues, so it is NOT negative definite.

(c) A 3×3 matrix A for which the eigenspaces satisfy

$$E_9 = \text{span}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \quad \text{and} \quad E_0 = (E_9)^\perp,$$

where $(E_9)^\perp$ is the orthogonal complement of E_9. (a plane)

A can be symmetric, by THM D23.3.

$E_9 = \ker(A - 9I_3) = \text{span}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$E_0 = \ker(A) = (E_9)^\perp$

Let A be symmetric; we use the spectral Theorem to find an orthogonal diagonalization for A. $S^{-1}AS = D \Rightarrow A = SDS^{-1}$, where S is ~~...~~ orthogonal and D is diagonal. Let $\lambda_1 = 9$, $\lambda_2 = 0$ with multiplicity 2. Then, $D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. To find S, note that $S = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}$, so we start by finding $(\vec{u}_2, \vec{u}_3)$ that form an ONB of E_0.

Using $E_0 = (E_9)^\perp$, we know: $\vec{u}_2, \vec{u}_3$ lie in the plane $x + 2y - 2z = 0$. Choose 2 $\perp$ vectors $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Then, $\vec{u}_2 = \frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$

$\vec{u}_3 = \frac{1}{\|\vec{v}_3\|}\vec{v}_3 = \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$.

$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$

$\therefore S = \begin{bmatrix} 1/3 & -2/\sqrt{5} & 2/\sqrt{5} \\ 2/3 & 1/\sqrt{5} & 0 \\ -2/3 & 0 & 1/\sqrt{5} \end{bmatrix}$ and $A = \begin{bmatrix} 1/3 & -2/\sqrt{5} & 2/\sqrt{5} \\ 2/3 & 1/\sqrt{5} & 0 \\ -2/3 & 0 & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/3 & -2/\sqrt{5} & 2/\sqrt{5} \\ 2/3 & 1/\sqrt{5} & 0 \\ -2/3 & 0 & 1/\sqrt{5} \end{bmatrix}^{-1}$

$$\boxed{A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix}}$$ (computed using technology)

This satisfies the given conditions because the eigenvalues of A are 0 w/ multiplicity 2, and 9 with multiplicity 1. This matches the dimensions of the given eigenspaces. Furthermore, A is symmetric and has 2 distinct eigenvalues, so the associated eigenspaces must be orthogonal, satisfying the given conditions.

Q6. (9 points) Determine whether each of the following statements is TRUE (meaning, "always true") or FALSE (meaning, "not always true") and give a justification.

(a) If an $n \times n$ matrix A is diagonalizable, then A^{T} is also diagonalizable.

$\boxed{\text{TRUE.}}$ If A is diagonalizable, then there exists an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$. Since the transpose of a diagonal matrix is diagonal, we can take the transpose of both sides:

$$(S^{-1}AS)^T = (D)^T$$
$$S^T A^T (S^{-1})^T = D^T = D \quad \text{using THM D14.7}$$
$$S^T A^T (S^T)^{-1} = D$$

Let $C = (S^T)^{-1}$, then $S^T = C^{-1}$ and $(S^T)^{-1} = C$.

Thus, we have $C^{-1}A^T C = D$ for an invertible matrix C and a diagonal matrix D. $\therefore$ A^T is diagonalizable.

Note that C is invertible b/c $C = (S^T)^{-1}$, and S^T is invertible b/c S is invertible. ($\det(S) = \det(S^T)$).

(b) If A and B are $n \times n$ positive semidefinite matrices, then $A + B$ is also positive semidefinite.

By Def 8.2.3: $q(\vec{x}) = \vec{x}^T A \vec{x}$, $q(\vec{x}) = \vec{x}^T B \vec{x}$ both ≥ 0 for all nonzero $\vec{x}$.

$$q(\vec{x}) = \vec{x}^T (A+B)\vec{x}$$
$$= \underbrace{\vec{x}^T A \vec{x}}_{\geq 0} + \underbrace{\vec{x}^T B \vec{x}}_{\geq 0} \geq 0.$$

$\boxed{\text{TRUE}}$, the sum of 2 non-negative values will always be non-negative (≥ 0), so $A+B$ must be positive semidefinite.

(c) If A is an $n \times n$ matrix and all of its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real numbers, then the singular values of A are given by $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$.

FALSE, this is only true if A is a symmetric matrix.

The singular values of A are the square roots of the eigenvalues of A^TA; $A^TA = A^2$ only when $A^T = A$, or by definition, when A is symmetric.

If $A^2 = A^TA$, then the eigenvalues of A^TA are the squares of the eigenvalues of A, so that the square roots of A^TA's eigenvalues, are A's singular values and coincide with A's eigenvalues. However, if A is not symmetric, this relationship does not hold.

ie: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has eigenvalues $\lambda_{A1} = 1$ w/ multiplicity $= 2$

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$f_{A^TA}(\lambda) = \lambda^2 - 3\lambda - 1$$

so $\lambda_1 = \frac{3+\sqrt{5}}{2}$, $\lambda_2 = \frac{3-\sqrt{5}}{2}$

and $\sigma_1 = \sqrt{\frac{3+\sqrt{5}}{2}}$, $\sigma_2 = \sqrt{\frac{3-\sqrt{5}}{2}}$

clearly, σ_1 and σ_2 do not coincide with the absolute value of the eigenvalues of $A = |\lambda_{A_1}| = |1| = 1$

on the other hand, for symmetric A:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has $\lambda_{A1} = 2$, $\lambda_{A2} = 1$

$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ with $\lambda_1 = 4$, $\lambda_2 = 1$

so, $\sigma_1 = \sqrt{4} = 2$ and $\sigma_2 = \sqrt{1} = 1$,

which matches with $|\lambda_{A_1}| = \sigma_1 = 2$ and $|\lambda_{A_2}| = 1 = \sigma_2$