MATH 33A LECTURE 3
MIDTERM I

Please note: Show your work. Correct answers not accompanied by sufficent explanations will receive little or no credit (except on multiple-choice problems). Please call one of the proctors if you have any questions about a problem. No calculators, computers, PDAs, cell phones, or other devices will be permitted. *If you have a question about the grading or believe that a problem has been graded incorrectly, you must bring it to the attention of your professor within 2 weeks of the exam.*

By signing above I certify that I am the person whose name and student ID appears on this page.

Copyright © 2016 UC Regents/Dimitri Shlyakhtenko. No part of this exam may be reproduced by any means, including electronically.

Problem 1. (True/False, 1 pt each) Mark your answers by filling in the appropriate box next to each question (no explanations are necessary on this problem).

(a) T F Let A and B be 5×5 matrices. If $\ker A = \operatorname{im} B$ then $AB = 0$.

(b) T F If A is an invertible $n \times n$ matrix, then the rank of A is n.

(c) T F There exists a 3×3 matrix A for which $\operatorname{im} A = \mathbb{R}^3$.

(d) T F If A is an $n \times m$ matrix and $\ker A = 0$, then $m \leq n$.

(e) T F The matrix $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in reduced row echelon form.

$AA' = IAA' \quad AA = A'$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(f) T F If $A \cdot A \cdot A \cdot A$ is the identity matrix, ~~then~~ A is invertible.

$\boxed{AAA} = A'$

(g) T F If $\ker A = \operatorname{im} A$ for some square $n \times n$ matrix A, then n must be even.

(h) T F The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is invertible. $\quad 1 \cdot 4 - 3 \cdot 2$

(i) T F If T is a counterclockwise rotation by 2.016 radians in the plane, then $\ker T = (0)$.

(j) T F The set $\{(x, y) : xy = 1\}$ is not a subspace of $\mathbb{R}^2$.

$xy = 1$

$(1,1)$

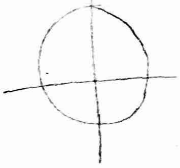
$(-1,-1)$

addition

subtraction

0

2.016



not closed under ~~addition~~ ~~invertible~~

$y = \frac{1}{x}$

$y+3 \neq \frac{1}{x+3}$

$2 = \frac{1}{2}$

$\frac{1}{2}$

$x = 2$

$y = \frac{1}{2}$

$x + 3 = 5$

$\frac{1}{2+3}$

$\frac{1}{2} + 5$

Problem 2. (10 pts) Let $A = \begin{bmatrix} 0 & 1 & 4 & 7 & 1 \\ 1 & 2 & 5 & 8 & 1 \\ 1 & 3 & 6 & 9 & 2 \end{bmatrix}$.

(a) Are the columns of A linearly independent?

no.

Col 2 = col 1 + col 5

6

(b) Find a basis for the image of A.

col's 1, 3, 4, 5 are lin independent

why?

$\cancel{\boxed{\text{basis } Im(A) = \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right)}}$

-4

(c) Find a basis for the kernel of A.

(3) III − I: 0 1 1 1 1

(2) II − I: 1 1 1 1 0

I − (3): 0 0 3 6 0

(4) = 0 0 1 2 0

(2) (3): 1 0 0 0 −1

(3) − (4): 0 1 0 −1 1

$\therefore x_1 = -x_2 - x_3 - x_4$

$x_2 = -x_3 - x_4 - x_5 - x_6$

$x_1 = x_5$

$x_3 = -2x_4$

$x_2 = x_4 - x_5$

let $x_4 = s$, $x_5 = t$

$$s\begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\boxed{\ker(A) = \left[\begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right]}$

Problem 3. (10 pts) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation whose matrix is given by:

$$A = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}.$$

$\frac{1}{4} \cdot \frac{3}{4} - \frac{\sqrt{3}}{4} \cdot \frac{\sqrt{3}}{4}$ = non-inv.

(a) Find $\ker A$.

$A = \begin{pmatrix} 1 & \sqrt{3} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}$ b $\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} \to \ker A = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$

let $b = x_2$

not lin. independent ∴ $x_1 = -\sqrt{3}\, x_2$

(b) Find $\operatorname{im} A$.

$\operatorname{im}(A) = \begin{pmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix}$

b/c col 2 = col 1 × $\sqrt{3}$.

~~(c)~~ Describe the transformation S geometrically.

$\begin{pmatrix} \frac{?}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

6

10

Problem 4. (10 pts) Find an invertible 2×2 matrix A so that $A \cdot A = A^{-1}$ (in other words, $A^2 = A^{-1}$). Hint: try to find the corresponding transformation first.

$A \cdot A = A^{-1}$ $\quad A^2 \cdot A^{-1} = I$

$A \cdot A \cdot A = I$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ cond: $ad - bc \neq 0$ invertible

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad a^2 + bc = e \quad ab + bd = f$

$ac + cd = g \quad db + d^2 = h$

cond $AA \cdot A^{-1} = I$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ s.t. $\quad ae + bg = 1 \quad af + bh = 0$

$ce + dg = 0 \quad cf + dh = 1$

$a^3 + abc + \cancel{bac} + \cancel{bcd} = 1 = \cancel{cab} + \cancel{cbd} + d^2b + d^3$

$ca^2 + bc^2 + dac + cd^2 = 0 = a^2b + abd + b^2d + bd^2$.

something that satisfies this $a, b, c, d \in \mathbb{R}$. $a = d = 1$ $c = b = 0$.

explaination: actually...

and the fact that $A \cdot A = I$, means that $A = A^{-1}$, and A is invertible.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

it's ok to find a matrix s.t. $A^2 = A$ & $A^2 = I$ $\Rightarrow$ $A \cdot A = A^{-1}$ & $A^2 \cdot A = I$

this includes $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is an identity matrix in itself, so $A \cdot A = I$ and $A \cdot A = A = I = A \cdot A^{-1} =$

Problem 5. (10 pts) Let P be the plane $x+y+z=0$ and Q be the plane $x+2y+3z=0$. Their intersection is a subspace which is a line. Find a basis for that subspace.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 0$$

Need to consider $\ker\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$.

$$P \cap Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 0$$

2

a	b	c	d	e	f
1	0	0	1	0	0
0	1	0	0	2	0
0	0	1	0	0	3

$a = -d$

$b = -2e$

$c = -3f$

$$d\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + e\begin{pmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + f\begin{pmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\ker(P \cap Q) = \left[\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right]$$