Math 33A: Midterm 1

2021 Spring

Instructions:

- ☐ The exam will begin on April 19th at 8AM PT. You will be given **24 hours** to complete and submit your works. The submission window will be closed on April 20th at 8AM.
- ☐ **No late submission** will be considered. Make sure to spare enough time to complete and submit your solutions. Make-ups for the exam are permitted only under exceptional circumstances, as outlined in the UCLA student handbook.
- ☐ The exam will be **open book/open notes**. You can use any resources you find in our textbook or on our CCLE page.
- ☐ You must **show your works to receive credit.** Partial credit will be scarce for incomplete solutions, so make sure to get everything right.
- ☐ You may use technology to write up your solutions, such as word processors or note-taking applications. You may also write your solutions on blank papers. If you choose to do so, please leave enough space between questions.
- ☐ A Gradescope link for submitting your work will be provided on the CCLE course webpage.
- ☐ If you have a question about the phrasing of the questions or about the exam logistics, you may email me (sos440@math.ucla.edu). Please make sure to begin the subject line of your email with the prefix 'Math 33A'; otherwise I will not reply to the email.
- ☐ You must **sign the code of conduct.** Any deviation from the rules will be considered as cheating. The university is also well-aware of "academic educational sites", and their use in connection with the exam is an Honor Code violation that is taken very seriously in UCLA.

Please read and sign the following honor code:

> *"I certify, on my honor, that I have not asked for or received assistance of any kind from any other person while working on the exam and that I have not used any non-permitted materials or technologies during the period of this evaluation."*

Name: [redacted] ______________________

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Q1. (10 points) Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 8 \\ 3 \\ k \end{bmatrix},$$

where k is a number.

(a) Determine the value of k such that $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

Using definition 1.3.9 of a linear combination:

$$\vec{v}_3 = x_1\vec{v}_1 + x_2\vec{v}_2$$

$$\begin{bmatrix} 8 \\ 3 \\ k \end{bmatrix} = x_1\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4x_1 + 3x_2 \\ x_1 + x_2 \\ -x_1 + x_2 \end{bmatrix}$$

It's augmented matrix is A; we reduce to rref to find k.

$$A = \left[\begin{array}{cc|c} 4 & 3 & 8 \\ 1 & 1 & 3 \\ -1 & 1 & k \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 4 & 3 & 8 \\ -1 & 1 & k \end{array}\right] \begin{matrix} \\ -4R_1 \\ +R_1 \end{matrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 2 & k+3 \end{array}\right] \begin{matrix} +R_2 \\ \times(-1) \\ \end{matrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 2 & k+3 \end{array}\right] \begin{matrix} \\ \\ -2R_2 \end{matrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & k-5 \end{array}\right]$$

using rref([A]), we see that $0 = k-5$, or $\boxed{k = 5}$ such that $\vec{v}_3 = x_1\vec{v}_1 + x_2\vec{v}_2$.

(b) Let k be as in the previous part. Find all solutions of the equation

$(k = 5)$

Note: so $\vec{v}_3 = \begin{bmatrix} 8 \\ 3 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$$

and then write your solution in parametric form.

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = x_1\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 8 \\ 3 \\ 5 \end{bmatrix}$$

The augmented matrix is: $A = \left[\begin{array}{ccc|c} 4 & 3 & 8 & 2 \\ 1 & 1 & 3 & 1 \\ -1 & 1 & 5 & 3 \end{array}\right]$. we compute rref(A):

(swap R_2 and R_1):

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 4 & 3 & 8 & 2 \\ -1 & 1 & 5 & 3 \end{array}\right] \begin{matrix} \\ -4R_1 \\ +R_1 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & -1 & -4 & -2 \\ 0 & 2 & 8 & 4 \end{array}\right] \begin{matrix} +R_2 \\ \times(-1) \\ +2R_2 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so, we have $\begin{cases} x_1 = x_3 - 1 \\ x_2 = -4x_3 + 2 \\ x_3 \text{ is free} \end{cases}$. parametrize the set of solutions using $-\infty < t < \infty$:

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}$$

Q2. (10 points) In computer graphics, we are often interested in finding a curve that interpolates a given set of points in a reasonably smooth way. For instance, consider the situation where we are given two line segments in $\mathbb{R}^2$ with known slopes and endpoints as in the figure below, and suppose we want to find a curve joining them.

One method often employed in this problem is to find a polynomial

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

such that

$$\left|\begin{array}{c} f(0) = y_1 \\ f(1) = y_2 \\ f'(0) = m_1 \\ f'(1) = m_2 \end{array}\right|, \qquad (\diamond)$$

where y_1, y_2, m_1, m_2 are numbers and $f'(t) = a_1 + 2a_2 t + 3a_3 t^2$ is the derivative of $f(t)$.

(a) By regarding a_0, a_1, a_2, a_3 as variables, write down the linear system $(\diamond)$ in matrix form.

$$\begin{cases} f(0) = a_0 + 0 + 0 + 0 = y_1 \\ f(1) = a_0 + a_1 + a_2 + a_3 = y_2 \\ f'(0) = 0 + a_1 + 0 + 0 = m_1 \\ f'(1) = 0 + a_1 + 2a_2 + 3a_3 = m_2 \end{cases}$$

using theorem 1.3.11, the matrix form of a linear system is $A\vec{x} = \vec{b}$, where A is the coefficient matrix.

$$\boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ m_1 \\ m_2 \end{bmatrix}}$$

is the linear system in matrix form.

(b) Solve the system ($\diamond$) for a_0, a_1, a_2, a_3. In other words, determine the formulas for a_0, a_1, a_2, a_3 in terms of y_1, y_2, m_1, m_2.

use the augmented matrix and apply Gauss - Jordan Elimination :

1. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & y_1 \\ 1 & 1 & 1 & 1 & y_2 \\ 0 & 1 & 0 & 0 & m_1 \\ 0 & 1 & 2 & 3 & m_2 \end{array}\right] \Rightarrow$ 2. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & y_1 \\ 0 & 1 & 0 & 0 & m_1 \\ 1 & 1 & 1 & 1 & y_2 \\ 0 & 1 & 2 & 3 & m_2 \end{array}\right] \begin{array}{l} \\ \\ -R_1-R_2 \\ -R_2 \end{array} \Rightarrow$ 3. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & y_1 \\ 0 & 1 & 0 & 0 & m_1 \\ 0 & 0 & 1 & 1 & y_2-y_1-m_1 \\ 0 & 0 & 2 & 3 & m_2-m_1 \end{array}\right]$

4. $(R_4 - 2R_3)$ $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & y_1 \\ 0 & 1 & 0 & 0 & m_1 \\ 0 & 0 & 1 & 1 & y_2-y_1-m_1 \\ 0 & 0 & 0 & 1 & m_2+m_1-2y_2+2y_1 \end{array}\right] \begin{array}{l} \\ \\ -R_4 \\ \\ \end{array} \Rightarrow$ 5. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & y_1 \\ 0 & 1 & 0 & 0 & m_1 \\ 0 & 0 & 1 & 0 & 3y_2-3y_1-2m_1-m_2 \\ 0 & 0 & 0 & 1 & m_2+m_1-2y_2+2y_1 \end{array}\right]$

so, we have :

$$\boxed{\begin{aligned} a_0 &= y_1 \\ a_1 &= m_1 \\ a_2 &= -3y_1+3y_2-2m_1-m_2 \\ a_3 &= 2y_1-2y_2+m_1+m_2 \end{aligned}}$$

(c) Find a polynomial $f(t)$ of degree at most 3 such that

$$f(0) = 2, \qquad f(1) = 2, \qquad f'(0) = 3, \qquad f'(1) = -2.$$

using part A, we know that $y_1 = 2;\ y_2 = 2;\ m_1 = 3;\ m_2 = -2$.

using the answer obtained in part B, we have

$a_0 = 2$, $a_1 = 3$, $a_2 = -3(2)+3(2)-2(3)-(-2)$
$= -6+2 = -4$

$a_3 = 2(2)-2(2)+3+(-2) = 1$

so, $\boxed{f(t) = 2 + 3t - 4t^2 + t^3}$

Q3. (10 points) Consider the following linear transformations:

- $T : \mathbb{R}^2 \to \mathbb{R}^2$ reflects any vector about the line $y = x$. $\to T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $S : \mathbb{R}^2 \to \mathbb{R}^2$ rotates any vector through an angle of 45° in the counter-clockwise direction.

(a) Find the matrices of T and S, respectively.
(That is, find the matrix A such that $T(\vec{x}) = A\vec{x}$ and the matrix B such that $S(\vec{x}) = B\vec{x}$.)

use theorem 2.1.2 and examine the standard vectors $\vec{e}_1$ and $\vec{e}_2$:

$T(\vec{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

so, $A = \begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix}$

$\boxed{A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$

$S(\vec{e}_1) = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$S(\vec{e}_2) = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

so, $B = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, or equivalently,

$B = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix}$ using theorem 2.2.3.

(b) Find the matrix C of the composition $T \circ S \circ T$.
(That is, find the matrix C such that $(T \circ S \circ T)(\vec{x}) = C\vec{x}$.)

$T \circ S \circ T \to T(S(T(\vec{x}))) = A(B(A\vec{x}))$, so $C = ABA$

$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

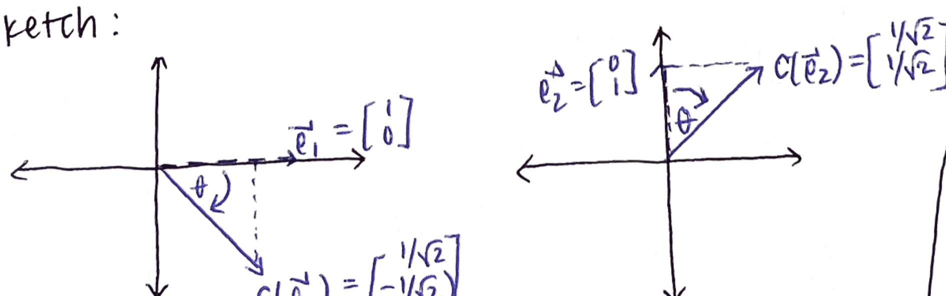
$(AB)A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = C}$

where we used the associative property (thm 2.3.6) of matrix multiplication and the definition of the composition of linear transformations.

(c) You are given that the composition $T \circ S \circ T$ reduces to one of the geometric transformations discussed in class. Interpret the transformation $T \circ S \circ T$ geometrically.

Observe $T(S(T(\vec{e}_1)))$ and $T(S(T(\vec{e}_2)))$. using part B, $C = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, and applying theorem 2.1.2, we know that $(T \circ S \circ T)(\vec{e}_1) = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ and $(T \circ S \circ T)(\vec{e}_2) = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

sketch:



so, the transformation $T \circ S \circ T$ is a rotation through an angle of 45° in the clockwise direction. (using inspection).

Q4. (10 points) Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that

$$T\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}1\\-2\\7\end{bmatrix} \quad \text{and} \quad T\begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}0\\4\\6\end{bmatrix}.$$

Let A be the matrix of T, i.e., A is the matrix such that $T(\vec{x}) = A\vec{x}$.

(a) Compute $T\begin{bmatrix}8\\3\end{bmatrix}$.

Use theorem 2.1.3: $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$

where $\vec{v} = \begin{bmatrix}3\\1\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}5\\2\end{bmatrix}$.

$\vec{v}+\vec{w} = \begin{bmatrix}3+5\\1+2\end{bmatrix} = \begin{bmatrix}8\\3\end{bmatrix}$, so

$$T(\vec{v}+\vec{w}) = T\left(\begin{bmatrix}8\\3\end{bmatrix}\right) = T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}5\\2\end{bmatrix}\right)$$

$$= \begin{bmatrix}1\\-2\\7\end{bmatrix} + \begin{bmatrix}0\\4\\6\end{bmatrix} = \begin{bmatrix}1+0\\-2+4\\7+6\end{bmatrix}$$

$$\boxed{T\left(\begin{bmatrix}8\\3\end{bmatrix}\right) = \begin{bmatrix}1\\2\\13\end{bmatrix}}$$

(b) What is the matrix product $A\begin{bmatrix}3 & 5\\1 & 2\end{bmatrix}$? Briefly explain why.

Using theorem 2.3.2, and let $\vec{v}_1 = \begin{bmatrix}3\\1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}5\\2\end{bmatrix}$,

we have:

$$A\begin{bmatrix}3 & 5\\1 & 2\end{bmatrix} = A\begin{bmatrix}| & |\\ \vec{v}_1 & \vec{v}_2\\ | & |\end{bmatrix} = \begin{bmatrix}| & |\\ A\vec{v}_1 & A\vec{v}_2\\ | & |\end{bmatrix}$$

Since A is the matrix of T, $A\vec{v}_1$ is simply $T(\vec{v}_1)$.

$A\vec{v}_2$ is $T(\vec{v}_2)$ by the same reasoning. So,

$$A\begin{bmatrix}3 & 5\\1 & 2\end{bmatrix} = \begin{bmatrix}| & |\\ T(\vec{v}_1) & T(\vec{v}_2)\\ | & |\end{bmatrix} = \begin{bmatrix}| & |\\ T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) & T\left(\begin{bmatrix}5\\2\end{bmatrix}\right)\\ | & |\end{bmatrix}$$

From the given information, $T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-2\\7\end{bmatrix}$ and

$T\left(\begin{bmatrix}5\\2\end{bmatrix}\right) = \begin{bmatrix}0\\4\\6\end{bmatrix}$, so $\boxed{A\begin{bmatrix}3 & 5\\1 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\-2 & 4\\7 & 6\end{bmatrix}}$

(c) Determine the matrix A.

Define the size of A: 3 x 2. 2 columns for product to be well defined; 3 rows because $T(\vec{x})$ is a 3 x 1 matrix.

Let $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Then,

$$T\begin{bmatrix} 3 \\ 1 \end{bmatrix} = A\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3a+b \\ 3c+d \\ 3e+f \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$$

$$T\begin{bmatrix} 5 \\ 2 \end{bmatrix} = A\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5a+2b \\ 5c+2d \\ 5e+2f \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}$$

Use an augmented matrix and Gauss-Jordan Elimination for every row of A:

$[a \ b]$: $\left[\begin{array}{cc|c} 3 & 1 & 1 \\ 5 & 2 & 0 \end{array}\right] \times(\frac{1}{3}) \Rightarrow \left[\begin{array}{cc|c} 1 & 1/3 & 1/3 \\ 5 & 2 & 0 \end{array}\right] -5R_1 \Rightarrow \left[\begin{array}{cc|c} 1 & 1/3 & 1/3 \\ 0 & 1/3 & -5/3 \end{array}\right] -R_2$

$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1/3 & -5/3 \end{array}\right] \times 3 \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -5 \end{array}\right]$ so $a = 2$, $b = -5$

$[c \ d]$: $\left[\begin{array}{cc|c} 3 & 1 & -2 \\ 5 & 2 & 4 \end{array}\right] \times(\frac{1}{3}) \Rightarrow \left[\begin{array}{cc|c} 1 & 1/3 & -2/3 \\ 5 & 2 & 4 \end{array}\right] -5R_1 \Rightarrow \left[\begin{array}{cc|c} 1 & 1/3 & -2/3 \\ 0 & 1/3 & 22/3 \end{array}\right] -R_2$

$\left[\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1/3 & 22/3 \end{array}\right] \times 3 \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1 & 22 \end{array}\right]$ so $c = -8$, $d = 22$

$[e \ f]$: $\left[\begin{array}{cc|c} 3 & 1 & 7 \\ 5 & 2 & 6 \end{array}\right] \times(\frac{1}{3}) \Rightarrow \left[\begin{array}{cc|c} 1 & 1/3 & 7/3 \\ 5 & 2 & 6 \end{array}\right] -5R_1 \Rightarrow \left[\begin{array}{cc|c} 1 & 1/3 & 7/3 \\ 0 & 1/3 & -17/3 \end{array}\right] -R_2$

$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1/3 & -17/3 \end{bmatrix} \times 3 \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & -17 \end{array}\right]$ so $e = 8$, $f = -17$

$$\therefore \boxed{A = \begin{bmatrix} 2 & -5 \\ -8 & 22 \\ 8 & -17 \end{bmatrix}}$$

Q5. (15 points) For each of the following problems, determine whether a matrix with the given properties exists or not. If such a matrix exists, then provide an example and briefly explain why your choice of matrix has the desired properties. Otherwise, use the results from class to explain why there is no such matrix.

(a) A 3×2 matrix A such that the system $A\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ has a unique solution.

$\boxed{\text{Yes}}$. Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then, $A\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

which has augmented matrix $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] = B$.

$\text{rref}(B) = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$, which has rank $= 2$. By theorem 1.3.1, there is a unique solution because all the variables are leading variables. In this example, $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) A 2×2 matrix A such that $A\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$.

$\boxed{\text{NO.}}$ If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} a+2b & 3a+6b \\ c+2d & 3c+6d \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$

The equations $\begin{cases} a+2b = 1 \\ 3a+6b = -1 \end{cases}$ and $\begin{cases} c+2d = 2 \\ 3c+6d = -2 \end{cases}$ are inconsistent.

Demonstrate using Gauss-Jordan Elimination for a, b:

$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 6 & -1 \end{array}\right] \begin{array}{c} \\ -3R_1 \end{array} \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -4 \end{array}\right]$ since $0 \neq -4$, there is no solution.

The equations for c, d are inconsistent for the same reason. Therefore, there is no matrix A that satisfies the given condition.

(c) A 2×2 matrix A such that $A \neq I_2$ but $A^5 = I_2$.
(Here, $A^5 = AAAAA$ denotes the product of five A's, and I_n is the $n \times n$ identity matrix.)

$\boxed{\text{Yes.}}$ Example: $A = \begin{bmatrix} \cos(72^\circ) & -\sin(72^\circ) \\ \sin(72^\circ) & \cos(72^\circ) \end{bmatrix}$, representing a counterclockwise rotation through a 72° angle. $A^5 = I_2$ because if A is interpreted as the matrix for a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$, then A^5 is simply $T(T(T(T(T(\vec{x}))))) = AAAAA\vec{x}$. This is the equivalent of applying the transformation 5 times. ~~so, in order for a linear transformation~~ so, $A^5 = I_2$ seeks a matrix A such that applying the linear transformation 5 times will ~~yeild~~ yield the original vector $\vec{x}$. Rotating C.C.W by 72° achieves this, as $72^\circ \times 5 = 360^\circ \Rightarrow$ full rotation back to original.

(d) An invertible 2×2 matrix A such that $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} A$ is also invertible.

$\boxed{\text{NO.}}$ $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} A$ will never be invertible. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then, $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+3c & b+3d \\ 3a+9c & 3b+9d \end{bmatrix} = M$.

Apply theorem 2.4.9, criteria for a 2x2 matrix to be invertible: $\det(M) \neq 0$.

$$\det(M) = (a+3c)(3b+9d) - (3a+9c)(b+3d)$$
$$= \cancel{3ab} + \cancel{9ad} + \cancel{9bc} + \cancel{27cd} - \cancel{3ab} - \cancel{9ad} - \cancel{9bc} - \cancel{27cd}$$
$$= 0$$

So, since the determinant of the product always $= 0$ for all matrices A, the product will never be invertible. This can also be concluded by noting that $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ is not invertible.

(e) A 2×2 matrix A such that $kI_2 - A$ is invertible for all k in $\mathbb{R}$.

$\boxed{\text{Yes.}}$ Example: $\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$.

Then, $kI_2 - A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} k-1 & -1 \\ 2 & k \end{bmatrix}$

$$\det(kI_2 - A) = k(k-1) - (2)(-1)$$
$$= k^2 - k - (-2) = k^2 - k + 2$$

$kI_2 - A$ is invertible when it's determinant does NOT equal 0 (by thm 2.4.9). Since $k^2 - k + 2 = 0$ has no real solution, $\det(kI_2 - A) = 0$ ~~has no~~ also has no real solution and thus it will never equal 0. Therefore, $\det(kI_2 - A) \neq 0$ for all real numbers k, so this matrix is invertible for all k in $\mathbb{R}$.