M33A Midterm 2: Version A

November 20, 2017

Name: [redacted]

Section: 3D

DO NOT OPEN THIS EXAM UNTIL TOLD TO DO SO.
JUSTIFY ALL ANSWERS UNLESS EXPLICITLY TOLD OTHERWISE.

Question	Points	Score
1	20	15
2	10	7
3	30	28
4	25	23
5	15	15
Total:	100	83

1. True or false, no justification required.

(a) (5 points) If A is a 3×3 matrix and $k \in \mathbb{R}$, then $\det(kA) = k^3 \det(A)$.

squre

$2\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

5 TRUE

5k

(b) (5 points) If the rows of an $n\times n$ matrix A form a basis of $\mathbb{R}^n$, then $\det(A) = 1$.

A^T cols = basis of $\mathbb{R}^n$

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ $\mathbb{R}^2$

$\det(A) = 9$

5 FALSE

(c) (5 points) For any 5×3 matrix A, the equation $A\vec{x} = \vec{b}$ has a *unique* least-squares solution.

A^TA

5 FALSE

det has a least squres solutn
but could have meny

(d) (5 points) For any matrix A, $\ker(A)^{\perp} = \operatorname{im}(A^{T})$.

$(\operatorname{im}(A))^{\perp} = \ker(A^T)$

$\operatorname{im}(A) = (\ker(A^T))^{\perp}$

$A \mapsto A^T$

0 FALSE

$\perp$ of both sides

2. (a) (5 points) Give an example of a 2×2 skew-symmetric orthogonal matrix.

$A^T = -A \quad A = -A^T$

orthonormal cols

presernes length

5 $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(b) (5 points) There are no 3×3 skew-symmetric orthogonal matrices. Why? *(Hint: Use determinants.)*

~~$-A = A^T$~~

$A^T = -A$

$k\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T \neq -\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$k a \cdot \det(\;) - k d \cdot \det(\;) + k g \det(\;)$

$(-1)^{\#\text{ of}}$

2

more

V is a plane in 3D space

$$\vec{v}_1 \quad \vec{v}_2$$

3. Let $V \subseteq \mathbb{R}^3$ be the subspace with basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

   (a) (15 points) Find an orthonormal basis $\mathcal{U}$ for V via the Gram-Schmidt process, starting with $\mathcal{B}$.

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \div \sqrt{2^2+2^2+1^2}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \div \sqrt{9}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \div 3 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \right) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1}{3} + \frac{2}{3} \right) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} \div \sqrt{\frac{2}{3}^2 + \frac{2}{3}^2 + \frac{1}{3}^2}$$

$$= \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} \div \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}}$$

$$= \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} \div 1 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$\boxed{\mathcal{U} = \left\{ \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \right\}}$$

15

(continued on next page)

(b) (10 points) Compute the orthogonal projection of $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$ to V. Write the result in $\mathcal{B}$ coordinates.

$$\text{proj}_V \vec{v} = (\vec{v} \cdot \vec{u}_1)\vec{u}_1 + (\vec{v} \cdot \vec{u}_2)\vec{u}_2$$

$$= \left(\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \right) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} + \left(\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} \right) \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$= \left(\frac{1}{3} - \frac{2}{3} + \frac{10}{3} \right) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} + \left(\frac{2}{3} + \frac{2}{3} + \frac{5}{3} \right) \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} + 3 \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \text{ in } \mathcal{U} \text{ coordinates}$$

$$[\cdot]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

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(c) (5 points) Complete the basis $\mathcal{U}$ from part (a) to an orthonormal basis for $\mathbb{R}^3$.

$$\mathcal{U} = \left\{ \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} \right\}$$

5

$$\text{ONB for } \mathbb{R}^3 = \left\{ \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} \right\}$$

double check!!!

$\vec{x}$

$\frac{4}{9} - \frac{2}{9} -$

$\frac{4}{9} - \frac{2}{9} - \frac{2}{9}$

$\frac{2}{9} + \frac{2}{9} - \frac{4}{9}$

$\frac{4}{9} - \frac{2}{9} + \frac{2}{9}$

$\frac{2}{9} + \frac{2}{9} + \frac{4}{9}$

4. (a) (5 points) If Q is a $m \times n$ matrix with orthonormal columns, what linear transformation does $Q^T Q$ represent? (No justification required.)

5 identity matrix

(b) (5 points) If Q is a $m \times n$ matrix with orthonormal columns, what linear transformation does QQ^T represent? (No justification required.)

3 the orthogonal projection on ... ?

(c) (15 points) Find the QR factorization of the matrix

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$$M = \begin{array}{c} \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \\ \begin{bmatrix} 0 & 3 & 4 \\ 1 & 1 & 0 \\ 0 & 4 & -3 \end{bmatrix} \end{array}.$$

To find Q, the ONB version of M...

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \div \sqrt{0^2+1^2+0^2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \div \sqrt{1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} - \left(\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} - (0+1+0) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \div \sqrt{9+16} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \div 5 = \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix}$$

$$\vec{v}_3 \cdot \vec{u}_2 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix} = \frac{12}{5} - \frac{12}{5} = 0$$

$$\vec{v}_3 \cdot \vec{u}_1 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} - 0 - 0 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$$

$$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} \div \sqrt{16+9} = \begin{bmatrix} 4/5 \\ 0 \\ -3/5 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 3/5 & 4/5 \\ 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \end{bmatrix}$$

$\vec{v}_2 \cdot \vec{u}_1$ $\vec{v}_3 \cdot \vec{u}_1$ $\vec{v}_3 \cdot \vec{v}_2$

$$R = \begin{bmatrix} \|v_1\| & \vec{v}_2 \cdot \vec{u}_1 & \vec{v}_3 \cdot \vec{u}_1 \\ 0 & \|\vec{v}_2^{\perp}\| & \vec{v}_3 \cdot \vec{u}_2 \\ 0 & 0 & \|\vec{v}_3^{\perp}\| \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

5. Let $A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(a) (5 points) Compute det(A). Is A invertible?

$\det(A) = ad - bc$

$= (-1)(2) - (2)(2)$

$= -2 - 4 = -6 \neq 0$ so yes invertible

5

(b) (10 points) Use the determinant to find all λ such that the matrix $A - \lambda I_2$ is not invertible. (Such a λ is called an *eigenvalue* of the matrix A.)

$\det\left(\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$

$= \det\left(\begin{bmatrix} -1-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix}\right)$

$= \cancel{\det}\,((-1-\lambda)(2-\lambda) - (2)(2)$

$= -2 + \lambda - 2\lambda + \lambda^2 - 4$

$= \lambda^2 - \lambda - 6 = 0$

$(\lambda + 2)(\lambda - 3) = 0$

$\boxed{\lambda = -2, 3}$

10

(c) (5 points (bonus)) Is A similar to a diagonal matrix? If so, which one? If not, why?

$B = S^{-1} A S$

is B diagonal?

$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$

0