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(1) Consider the following matrix and vector:

$$A = \begin{bmatrix} 1 & 2 & 4 & 5 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

(a) What is the general solution of the linear system $A\vec{x} = \vec{b}$.

(b) What is the row-reduced echelon form of A.

(c) Find an orthonormal basis for the kernel of A.

(d) Indicate an entry in A that is currently a 0 but if changed to the number 1, would give an invertible matrix.

(e) Find an alternate solution to part (d). (There was more than one correct answer.)

a) $\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 4 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 4 & 5 & 4 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & -5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 + x_4 = -5$
$x_2 = 0$
$x_3 + x_4 = 1/2$
$x_5 = 1$
$x_4 = *$

$\Rightarrow \boxed{\begin{bmatrix} -5 - x_4 \\ 0 \\ 1/2 - x_4 \\ x_4 \\ 1 \end{bmatrix}}$

b) $\boxed{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$

c) $x_1 + x_4 = 0$
$x_2 = 0$
$x_3 + x_4 = 0$
$x_5 = 0$
$x_4 = *$

$\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ $\boxed{\frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}$

d) change entry A_{54} to a 1 (5th row, 4th column)

e) change entry A_{53} to a 1 (5th row, 3rd column)

(2) Consider the following singular-value decomposition of a matrix A:

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 & -1/2 & -1/2 \\ 0 & 1/2 & -1/2 & 1/2 & -1/2 \\ 0 & 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma \;\; 5\times 2} \underbrace{\begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}}_{V^T}$$

(a) What are the rank and nullity of A?

(b) What are the shortest vectors $\vec{x}$ so that $\|A\vec{x}\| = 1$? List all answers.

(c) What is the characteristic polynomial of AA^T?

(d) Find the vector in $\mathrm{im}(A)$ that is nearest to the point

$$\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

a) Rank: 2

b) $\|U\Sigma V^T\vec{x}\| = 1$

$\|U\Sigma V^T\vec{x}\|^2 = 1 \Rightarrow \|\Sigma \underbrace{V^T\vec{x}}_{\vec{y}}\|^2 = 1$ $\qquad \vec{y} = V^T\vec{x}$, $\vec{x} = V\vec{y}$

$$\left\| \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{y} \right\|^2 = 1 \qquad 3y_1^2 + 2y_2^2 = 1$$

$3\times 2 \cdot 2\times 1$

$y_1 = 1 \quad y_2 = 0$

$3y_1^2 = 1$

$y_1^2 = \frac{1}{3} \qquad y_1 = \pm 1/\sqrt{3} \qquad \begin{bmatrix} \pm 1/\sqrt{3} \\ 0 \end{bmatrix}$

$\vec{x} = V\vec{y} = \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$

$$\boxed{\pm \frac{1}{5\sqrt{3}} \begin{bmatrix} 3 \\ -4 \end{bmatrix}}$$

c) $AA^T = U\Sigma V^T V \Sigma^T U^T = U\Sigma\Sigma^T U^T$

$$= U \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix} U^T = U \begin{bmatrix} 9 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} U^T$$

9, 4, 0 (mult 3)

$$\boxed{p(\lambda) = (\lambda - 9)(\lambda - 4)\lambda^3}$$

d) $\text{Im}(A) = \text{im}(U \Sigma V^T)$

$$= \text{im}(U\Sigma)$$

$$= \left\{ U \begin{bmatrix} 3z_1 \\ 2z_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \; z_1, z_2 \in \mathbb{R}$$

$$= \text{span}(\vec{u}_1, \vec{u}_2)$$

$$= \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right)$$

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \left(\begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

extra paper

$$= [1] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \left[\frac{3}{2} \right] \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \end{bmatrix}}$$

(3) All parts of this problem relate to the following matrix

$$A = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

where the parameter $t > 0$ is unspecified. (Your answers may be functions of t.)

(a) What type of geometric transformation does A describe?

(b) What are the algebraic and geometric multiplicities of the eigenvalue 1?

(c) What is the largest singular value of A?

(d) Give the QR factorization of A.

a) A represents a horizontal shear

b) $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -t \\ 0 & \lambda-1 \end{vmatrix}$ $\boxed{\text{algebraic mult. is } 2}$

$\begin{bmatrix} 0 & -t \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\boxed{\text{geometric mult. is } 1}$

c) $\boxed{1}$ eigenvalues: 1 $\sqrt{1} = 1$

d) $\frac{1}{\sqrt{1}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} t \\ 1 \end{bmatrix} - \left(\begin{bmatrix} t \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix} - \left(\left[\, t \,\right] \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$\begin{bmatrix} t \\ 1 \end{bmatrix} - \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $R = Q^T A$ $Q^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

$$\boxed{\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q}\underbrace{\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}}_{R} = A = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}}$$

(4) (a) Let V be a subspace of $\mathbb{R}^n$. Explain why it is *never* correct to say

$$\text{span}(\vec{v}_1, \ldots, \vec{v}_k) \text{ is a basis for } V.$$

(b) Give an example of a 4×4 matrix that is both symmetric and in row-reduced echelon form.

(c) Let A and B be 5×5 matrices. Suppose the columns of A are linearly independent. Show that

$$\text{nullity}(AB) = \text{nullity}(BA)$$

a) The span of $(\vec{v}_1, \ldots, \vec{v}_k)$ contains all of the linear combinations of $\vec{v}_1, \ldots, \vec{v}_k$, and these aren't linearly independent
A basis must consist of linearly independent vectors
so a span of vectors can never be a basis

b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c) $A \Rightarrow \text{nullity}(A) = 0$ because columns of A are linearly independent

$5 = \dim(\ker(AB) + \dim(\text{im}(A))$

A is invertible $\Rightarrow A^{-1}A = I$

$\vec{x} \in \text{nullity}(AB)$

$AB\vec{x} = 0 \quad A^{-1}AB\vec{x} = A^{-1}0 \quad IB\vec{x} = 0 \quad B\vec{x} = 0$

$x \in \text{nullity}(B)$

$\vec{z} \in \text{nullity}(B)$

$B\vec{y} = 0 \Rightarrow AB\vec{y} = A0 = 0 \quad y \in \text{nullity}(AB$

$\text{nullity}(B) \in \text{nullity}(AB)$

$\Rightarrow \text{nullity}(AB) = \text{nullity}(B)$

~~BA~~ $\vec{y} \in \text{nullity}(BA)$

$BA\vec{y} = 0$

$BAA^{-1}\vec{y} = A^{-1}0$

$B\vec{y} = 0 \quad \vec{y} \in \text{nullity}(B)$

$\vec{g} \in \text{nulity}(B)$

$B\vec{g} = 0 \Rightarrow BA\vec{g} = 0$

$\text{nulity}(B) \in \text{nulity}(BA) \Rightarrow \text{nullity}(BA) = \text{nullity}(B)$

$\therefore \text{nullity}(AB) = \text{nullity}(BA)$ ✓

(5) Consider the plane $2z = 3x + 6y$ living in $\mathbb{R}^3$. We call this plane P.

(a) Find the matrix representing the the orthogonal projection onto P (in the standard basis).

(b) Find the matrix representing the orthogonal projection onto P in the basis

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right).$$

(c) If I reflect the point $(1, 1, 1)$ through the plane P, where does it end up?

a) $3x + 6y - 2z = 0 \qquad n = \frac{1}{7}\begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}$

$$\vec{x}_\perp = \vec{n}\,\vec{n}^T\,\vec{x} = \frac{1}{49}\begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}\begin{bmatrix} 3 & 6 & -2 \end{bmatrix} = \frac{1}{49}\begin{bmatrix} 9 & 18 & -6 \\ 18 & 36 & -12 \\ -6 & -12 & 4 \end{bmatrix}$$

$$I - \vec{x}_\perp = \boxed{\frac{1}{49}\begin{bmatrix} 40 & -18 & 6 \\ -18 & 13 & 12 \\ 6 & 12 & 45 \end{bmatrix}}$$

b) $B = S^{-1}AS$

$$S^{-1} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$S^{-1}A = \frac{1}{49}\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 40 & -18 & 6 \\ -18 & 13 & 12 \\ 6 & 12 & 45 \end{bmatrix} = \frac{1}{49}\begin{bmatrix} 16 & -17 & -27 \\ 24 & -1 & 33 \\ -10 & 29 & 72 \end{bmatrix}$$

$$S^{-1}AS = \boxed{\frac{1}{49}\begin{bmatrix} -28 & 33 & -44 \\ 56 & 25 & 32 \\ 91 & -39 & 101 \end{bmatrix}}$$

c) $\vec{x} = [I - 2\vec{n}\vec{n}^T]\vec{x}$

$$= I - \frac{1}{49}\begin{bmatrix} 18 & 36 & -12 \\ 36 & 72 & -24 \\ -12 & 24 & 8 \end{bmatrix} = \frac{1}{49}\begin{bmatrix} 31 & -36 & 12 \\ -36 & -23 & 24 \\ 12 & 24 & 41 \end{bmatrix}$$

$$\vec{x}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \boxed{\frac{1}{7}\begin{bmatrix} 1 \\ -5 \\ 11 \end{bmatrix}}$$

(6) Consider the following matrix

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 5 \\ t & 0 & 0 & 4 \end{bmatrix}$$

which depends on the parameter t.

(a) Compute $\det(A)$. In the process, demonstrate the following techniques:

(i) At least one Laplace expansion (circle this; in color, if not too much trouble).

(ii) At least one row operation (box this; in color, if not too much trouble).

You are only required to compute $\det(A)$ once, provided you employ both techniques.

(b) For what values of t is A invertible?

(c) What is the top-right entry of A^{-1}, when t is such that A is actually invertible.

a) $\begin{vmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 5 \\ t & 0 & 0 & 4 \end{vmatrix} \begin{matrix} \\ \\ \\ -tR_1 \end{matrix} \Rightarrow \begin{vmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 5 \\ 0 & -3t & -5t & 4-7t \end{vmatrix} = 1 \times \det \begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ -3t & -5t & 4-7t \end{bmatrix}$

$= 1 \times \det \begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & t & 4+2t \end{bmatrix} = 1 \times 2 \times \det \begin{bmatrix} 3 & 5 \\ t & 4+2t \end{bmatrix}$

$+\frac{3t}{2}\Rightarrow$

$= 2 \times [(12+6t) - 5t)]$

$= 2 \times (12+t) = \boxed{24+2t}$

b) $24 + 2t \neq 0$

$\boxed{t \neq -12}$ $t \in \mathbb{R}$

c) $A(\vec{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$\frac{\det \begin{bmatrix} 0 & 3 & 5 & 7 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 5 \\ 1 & 0 & 0 & 4 \end{bmatrix}}{24+2t}$$

$-1 \times \det \begin{bmatrix} 3 & 5 & 7 \\ 2 & 4 & 6 \\ 0 & 3 & 5 \end{bmatrix}$

$= -1 \times -2 = 2$

$$\frac{2}{24+2t} = \boxed{\frac{1}{12+t}}$$

(7) In this problem we consider

$$A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \quad \text{and} \quad q(\vec{x}) = \vec{x} \cdot A\vec{x}$$

(a) Orthogonally diagonalize the matrix A. That is, find a diagonal matrix D and an orthogonal matrix S so that

$$A = SDS^T.$$

(b) In what directions ~~can I~~ travel from the origin maintaining $q(\vec{x}) = 0$? List all answers.

(c) What is the largest value that $q(\vec{x})$ takes on the circle $\|\vec{x}\| = 2$?

(d) What points on the circle $\|\vec{x}\| = 2$ yield these largest values? List all answers.

a) $\det(\lambda I - A) = \begin{bmatrix} \lambda - 9 & -12 \\ -12 & \lambda - 16 \end{bmatrix} = \lambda^2 - 25\lambda + 144 - 144$

$= \lambda(\lambda - 25)$

$\lambda = 0, \ \lambda = 25$

$\underline{\lambda = 25}$ $\begin{bmatrix} \cancel{16} \; 4 & -\cancel{12} \; 3 \\ -\cancel{12} \; -4 & \cancel{9} \; 3 \end{bmatrix}$ $\quad 4x_1 = 3x_2 \quad \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\underline{\lambda = 0}$ $\begin{bmatrix} -\cancel{9} \; -3 & -\cancel{12} \; -4 \\ -\cancel{12} \; -3 & -\cancel{16} \; -4 \end{bmatrix}$ $\quad -3x_1 = 4x_2$

$x_1 = -\frac{4}{3}x_2 \quad \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

$S = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4 5 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 25 \end{bmatrix}$

b) $q(\vec{x}) = \vec{x} \cdot A\vec{x} = 0 \quad \Rightarrow \quad \lambda_1 c_1^2 + \lambda_2 c_2^2 = 0$

$0c_1^2 + 25c_2^2 = 0$

$25c_2^2 = 0 \quad c_2 = 0$

$(c_1, 0)$

go along the line $\boxed{c_2 = 0}$

c) $\|\vec{x}\| = 2 \quad \sqrt{x_1^2 + x_2^2} = 2$

maximize $\lambda_2 \quad \lambda_2 = 25$

$x_1 = 0$

$\sqrt{0 + C_2^2} = 2$

$C_2 = \pm 2$

$0 C_1^2 + 25 C_2^2 = q(\vec{x})$

$= 25 C_2^2$

$25(2)^2 = \boxed{100}$

d) $\langle 0, 2 \rangle, \langle 0, -2 \rangle$

$$0 + 2 \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 6/5 \\ 8/5 \end{bmatrix}$$

$$0 \; -2 \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} -6/5 \\ -8/5 \end{bmatrix}$$

extra paper

(8) Consider the following matrix and vector:

$$A = \begin{bmatrix} 7 & -1 & 1 & 1 \\ -1 & 7 & 1 & 1 \\ 1 & 1 & 1 & 5 \\ 1 & 1 & 5 & 1 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Determine $A\vec{v}$ and $A^{-1}\vec{v}$.

(b) Determine the coordinates of $\vec{v}$ in the basis formed from the columns of A.

(c) The numbers 4 and 8 are eigenvalues of A. What are their geometric multiplicities? (Provide irrefutable justification.)

(c) What is the characteristic polynomial of A^{-1}?

a) $\boxed{A\vec{v} = \begin{bmatrix} 8 \\ 8 \\ 8 \\ 8 \end{bmatrix}}$

A is symmetric ∴ A is orth. diagonizable

$A = SDS^T \qquad A^{-1}A = I$

$SD^{-1}S^TSDS^T = I \qquad A^{-1} = SD^{-1}S^T$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-7 & 1 & -1 & -1 \\ 1 & \lambda-7 & -1 & -1 \\ -1 & -1 & \lambda-1 & -5 \\ -1 & -1 & -5 & \lambda-1 \end{vmatrix} \Rightarrow \begin{vmatrix} \lambda-7 & 1 & -1 & -1 \\ 0 & \lambda-8 & \lambda-2 & -6 \\ 0 & 0 & \lambda+4 & -\lambda-4 \\ -1 & -1 & -5 & \lambda-1 \end{vmatrix}$$

$$\Rightarrow (\lambda-1)\det\begin{bmatrix} \lambda-8 & \lambda-2 & -6 \\ 0 & \lambda+4 & -\lambda-4 \\ -1 & -5 & \lambda-1 \end{bmatrix}$$

$$+1 \det\begin{bmatrix} 1 & -1 & -1 \\ \lambda-7 & -1 & -1 \\ -1 & \lambda-1 & -5 \end{bmatrix}$$

$$\Rightarrow (\lambda-1)\Big[(\lambda-8)\big((\lambda+4)(\lambda-1) - (-5)(-\lambda-4)\big) - 1\big((\lambda-2)(-\lambda-4) - (-6)(\lambda+4)\big)\Big]$$

$$+ \big(-1 \times -5 - (-1)(\lambda-1)\big) + 1\big((\lambda-7)(-5) - (-1)(-1)\big] - 1 \big(+1 - (\lambda-7)(\lambda-1)\big)$$

$$= \lambda^4 - 16\lambda^3 + 48\lambda^2 + 256\lambda - 1024$$

$$= (\lambda-8)^2(\lambda+4)(\lambda-4)$$

$\lambda = 8, -4, 4$

$\underline{\lambda=4}$

rref $\begin{bmatrix} 11 & -1 & 1 & 1 \\ -1 & 11 & 1 & 1 \\ 1 & 1 & 5 & 5 \\ 1 & 1 & 5 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 11 & -1 & -1 & 1 \\ 0 & \frac{120}{11} & \frac{12}{11} & \frac{12}{11} \\ 0 & 12 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 11 & -1 & -1 & 1 \\ 0 & \frac{120}{11} & \frac{12}{11} & \frac{12}{11} \\ 0 & 0 & \frac{24}{5} & \frac{24}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 11 & -1 & 0 & 0 \\ 0 & \frac{120}{11} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$\underline{\lambda=4}$

$\begin{bmatrix} 3 & -1 & 1 & 1 \\ -1 & 3 & 1 & 1 \\ 1 & 1 & -3 & 5 \\ 1 & 1 & 5 & -3 \end{bmatrix} \quad \begin{bmatrix} 3 & -1 & 2 & 0 \\ -1 & 3 & 2 & 0 \\ 0 & 4 & -2 & 6 \\ 0 & 0 & \cancel{8} & -\cancel{8} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \quad \frac{1}{\sqrt{4}} \Rightarrow \frac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix},$

$\underline{\lambda=8}$

$\begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -7 & 5 \\ 1 & 1 & 5 & -7 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} -\cancel{1} \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$- \frac{1}{2} \; [-2$

$\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$A = \begin{pmatrix} 0 & -1/2 & -1/\sqrt{2} & 1/2 \\ 0 & -1/2 & 1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 1/2 & 0 & 1/2 \\ 1/\sqrt{2} & 1/2 & 0 & 1/2 \end{pmatrix} \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} -1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/8 & 0 \\ 0 & 0 & 0 & 1/8 \end{bmatrix}$$

$$S D^{-1} S^T = A^{-1} = \frac{1}{32} \begin{bmatrix} 5 & 1 & -1 & -1 \\ 1 & 5 & -1 & -1 \\ -1 & -1 & -1 & 7 \\ -1 & -1 & 7 & -1 \end{bmatrix}$$

$$\boxed{A^{-1} \vec{N} = \begin{bmatrix} 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \end{bmatrix}}$$

b) $\vec{v}_s = [S]\vec{x}_b$

$$\vec{x}_b = S^{-1}\vec{v}_s = \boxed{\begin{bmatrix} 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \end{bmatrix}}$$

c) as seen in part a, work * see part A work *

char poly of $A = p(\lambda) = \lambda^4 - 16\lambda^3 + 48\lambda^2 + 256 - 1024$

$= (\lambda-8)^2(\lambda+4)(\lambda-4)$

$\lambda = 8$ (alg mult 2), $-4, 4$ > 4 & 8 are eigen vectors

$\Rightarrow \underline{\lambda = 4}$ $\Rightarrow$ eigenspace = span $\left(\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}\right)$

$\therefore$ geom multiplicity 1

$\Rightarrow \underline{\lambda = 8}$ $\Rightarrow$ eigenspace = span $\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right)$

$\Rightarrow \therefore$ geom multipicity : 2

d) $A^{-1} = \begin{bmatrix} 0 & -1/2 & -1/\sqrt{2} & 1/2 \\ 0 & -1/2 & 1/\sqrt{2} & 1/2 \\ -1/2 & 1/2 & 0 & 1/2 \\ 1/\sqrt{2} & 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} -1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/8 & 0 \\ 0 & 0 & 0 & 1/8 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}$

$= SD^{-1}S^T$

eigen values: $-1/4,\ 1/4,\ 1/8,\ 1/8$

$$\boxed{\begin{aligned} p(\lambda) &= (\lambda - 1/4)(\lambda + 1/4)(\lambda - 1/8)^2 \\ &= x^4 - \frac{x^2}{4} - \frac{3x^2}{64} + \frac{x}{64} - \frac{1}{1024} \end{aligned}}$$